Travaux Dirigés de Physique Nucléaire et Physique des Particules

Shell model and applications

#1: Harmonic oscillator

1 / Explain why the energies of the 1D oscillator

$$h = \frac{P^2}{2\,\mu} + \frac{K}{2} X^2 \,,$$

are proportional to $\hbar\sqrt{K/\mu}$. From now on, the strength is denoted $K = \mu\,\omega^2$.

2 / More explicitly, show that

$$h = \frac{\hbar\,\omega}{2}\,\tilde{h} = \frac{\hbar\,\omega}{2}\left[-\frac{\mathrm{d}^2}{\mathrm{d}x^2} + x^2\right] \,,$$

where $\tilde{h}$ involves dimensionless quantities.

3 / Show that $\varphi_0(x) = \pi^{-1/4}\exp(-x^2/2)$ is the normalised ground-state of $\tilde{h}$ with eigenvalue $+1$, either directly from the Schrödinger operator or from the lowering operator $a = (\mathrm{d}/\mathrm{d}x + x)/\sqrt{2}$.

4 / Deduce the two next eigenvalues and eigenstates of $\tilde{h}$ using the raising operator $a^\dagger = (-\mathrm{d}/\mathrm{d}x + x)/\sqrt{2}$.

5 / (optional) Show that $\tilde{h}$ has eigenvalues $1 + 2\,n$, where $n = 0, 1, 2, \ldots$ and normalised eigenvectors

$$\left[2^n\, n!\,\sqrt{\pi}\right]^{-1/2} \exp(-x^2/2)\, H_n(x) \,,$$

where H_n is an Hermite polynomial,[1] or for n even or odd, respectively

$$\left[\frac{n!}{\Gamma(n+3/2)}\right]^{1/2} \exp(-x^2/2)\, L_n^{-1/2}(x^2)$$

$$\left[\frac{n!}{\Gamma(n+3/2)}\right]^{1/2} \exp(-x^2/2)\, x\, L_n^{1/2}(x^2) \,,$$

where L_n^β denotes an associated Laguerre polynomial, of which some useful properties are listed in Appendix.

6 / From the results on $\tilde{h}$, construct the two first s-wave levels and the first p-level of

$$H = \frac{\boldsymbol{p}^2}{2\,\mu} + \frac{\mu\,\omega^2}{2}\boldsymbol{r}^2 \,.$$

Indicate the degeneracy of each level.

7 / H is now treated in spherical coordinates. Show that the eigenfunctions can be seek as

$$\Phi_{n\ell m}(\boldsymbol{r}) = R_{n\ell}(r)\, Y_\ell^m(\theta, \varphi)$$

where $Y_\ell^m(\theta, \varphi)$ is the usual spherical harmonics. Indicate the meaning of the quantum numbers n, ℓ and m. Explain why the radial function does not depend on m.

8 / (optional) Write the radial Schrödinger equation obeyed by $R_{n\ell}(r)$ and deduce

$$R_{n\ell}(r) = A\,(\alpha r)^\ell\, \exp[-\alpha^2 r^2/2]\, L_n^{\ell+1/2}(\alpha^2 r^2) \,,$$

$$A = \left[\frac{2\,n!\,\alpha^3}{\Gamma(n+\ell+3/2)}\right]^{1/2} \,,$$

9 / Show that the same energy levels $\hbar\,\omega(N+3/2)$ are obtained as by the method of Cartesian coordinates, with the same degeneracy $d(N)$. Are the eigenfunctions the same by the two methods?

#2: Proton density of ^{16}O

1 / Show that for a spherical nucleus with closed shells, the proton density reads

$$\rho(r) = 2 \sum_{\substack{n,\ell \\ \text{occupied}}} \frac{2l+1}{4\,\pi} |R_{n\ell}(r)|^2$$

2 / Show that for the ground state of ^{16}O, with 2 protons in the $1s$ shell and 6 in $1p$, the charge density is

$$\rho(r) = \frac{2\,\alpha^3}{\pi^{3/2}}\,(1 + 2\alpha^2 r^2)\,\exp(-\alpha^2 r^2) \,.$$

Plot the graph of this function $\rho(r)$.

[1] The Hermite polynomials fulfill the orthogonality relation $\int_{-\infty}^{+\infty} \exp(-x^2) H_n(x) H_m(x) = \delta_{nm} 2^n n! \sqrt{\pi}$. $\exp(-x^2/2) H_n(x)$ obeys $y''(x) + (2n+1-x^2)y(x) = 0$.

#3: Energy levels, magic numbers

1 / Describe the energy spectrum corresponding to

- $V(r) = \mu\omega^2 r^2/2 + D\,\boldsymbol{\ell}^2$
- $V(r) = \mu\omega^2 r^2/2 + D\,\boldsymbol{\ell}^2 + g\,\boldsymbol{\ell}.\boldsymbol{s}$

starting from a pure oscillator with energy levels $E = \hbar\omega(2\,n + \ell + 3/2)$, using $D = -0.025\hbar\omega$ and $g = -0.05\hbar\omega$. Describe the degeneracy for the pure oscillator and the above perturbed oscillators. Deduce the first magic numbers.

2 / Predict the spin and parity for the ground state, first excited state and one of the next excited states of the following nuclei: ${}^{209}_{82}\mathrm{Pb}$, ${}^{209}_{83}\mathrm{Bi}$, ${}^{41}_{20}\mathrm{Ca}$. For both neutrons and protons, the shells are occupied in the following order:

$$\begin{array}{l} 1s_{1/2} \quad 1p_{3/2}\,1p_{1/2} \quad 1d_{5/2}\,2s_{1/2}\,1d_{3/2} \\ 1f_{7/2} \quad 2p_{3/2}\,1f_{5/2}2p_{1/2}\,1g_{9/2} \\ 2d_{5/2}\,1g_{7/2}\,1h_{11/2}\,2d_{3/2}\,3s_{1/2} \\ 1h_{9/2}\,2f_{7/2}\,2f_{5/2}\,3p_{3/2}\,1i_{13/2}\,3p_{1/2} \\ 2g_{9/2}\,1i_{11/2}\,3d_{5/2}\,2g_{7/2}\ldots \end{array}$$

#4: Schmidt lines

1 / Estimate the magnetic moment of a nucleus with an odd number A of nucleons, assuming it is due only to the last nucleon. The magnetic moment operator reads:

$$\boldsymbol{\mu} = \mu_N\,(g_s\,\boldsymbol{s} + g_\ell\,\boldsymbol{\ell})\ ,$$

with $g_{sp} = 5.59$, $g_{sn} = -3.82$, $g_{\ell p} = 1$ et $g_{\ell n} = 0$ in units of the nuclear magneton, $e\hbar/(2\,m_p) = 3.15\times 10^{-14}$ MeV/T, using the projection theorem for a vector operator $\boldsymbol{V}$:

$$\langle\alpha jm|V_k|\alpha jm\rangle = \frac{\langle\alpha jm|\boldsymbol{V}.\boldsymbol{J}|\alpha jm\rangle}{\langle\alpha jm|J^2|\alpha jm\rangle} \times \langle\alpha jm|J_k|\alpha jm\rangle$$

2 / Estimate the expectation value of $\boldsymbol{\mu}$ within the state $|jm\rangle$ with $m = j$.

3 / As an application, estimate the magnetic moment of the ground state of the nuclei ${}^{17}_{9}\mathrm{F}$ and ${}^{17}_{8}\mathrm{O}$.

Appendix: Laguerre polynomials

The associated Laguerre polynomials L_n^β are orthogonal polynomials, with increasing degree and increasing number of nodes, for the measure $\exp(-u)\,u^\beta$, in $u \in [0,\infty[$, with normalisation

$$\int_0^\infty \exp(-u)\,u^\beta\,L_n^\beta(u)\,L_m^\beta(u)\,\mathrm{d}u = \delta_{nm}\,\Gamma(\beta+n+1)/n!\ .$$

Some examples are $L_0^{1/2}(u) = L_0^{3/2}(u) = 1$ and $L_1^{1/2}(u) = 3/2 - u$.

They obey, among others, the following differential equation

$$u\,y''(u) + (\beta+1-u)\,y'(u) + n\,y(u) = 0\ ,$$

and it is straightforward to show that $\exp(-x^2/2)\,x^{\beta+1/2}\,L_n^\beta(x^2)$ obey

$$y''(x) + \left(4\,n + 2\,\beta + 2 - x^2 + \frac{1-4\,\beta^2}{4\,x^2}\right) y(x) = 0\ .$$

A generating function is

$$(1-z)^{-\beta-1}\,\exp[x\,z/(z-1)] = \sum_n L_n^\beta(x)\,z^n\ .$$