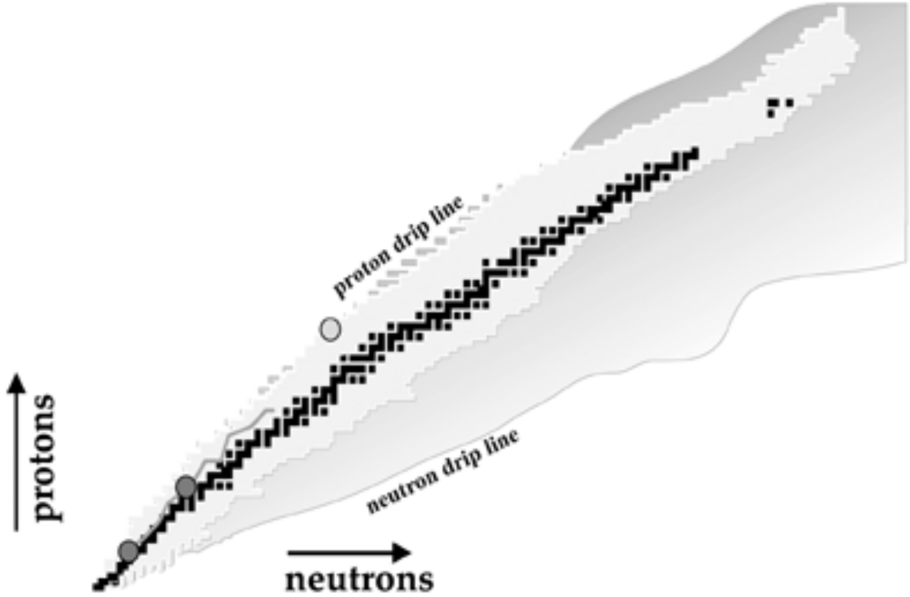
Chapter 4
The masses of the Nuclei



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    3. Coulomb term
    4. Asymmetry term
    5. Pairing term
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    3. α stability
    4. Fission
    5. Nucleus deformation
    6. Neutron star

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1- Experimental facts

1.1- Stable nuclei

(Z,N) chart of stable odd-A nuclei

- There is only one isobar for each value of A
- The line passing trough the average position of the dots on the (Z,N) plane is called the line of stability

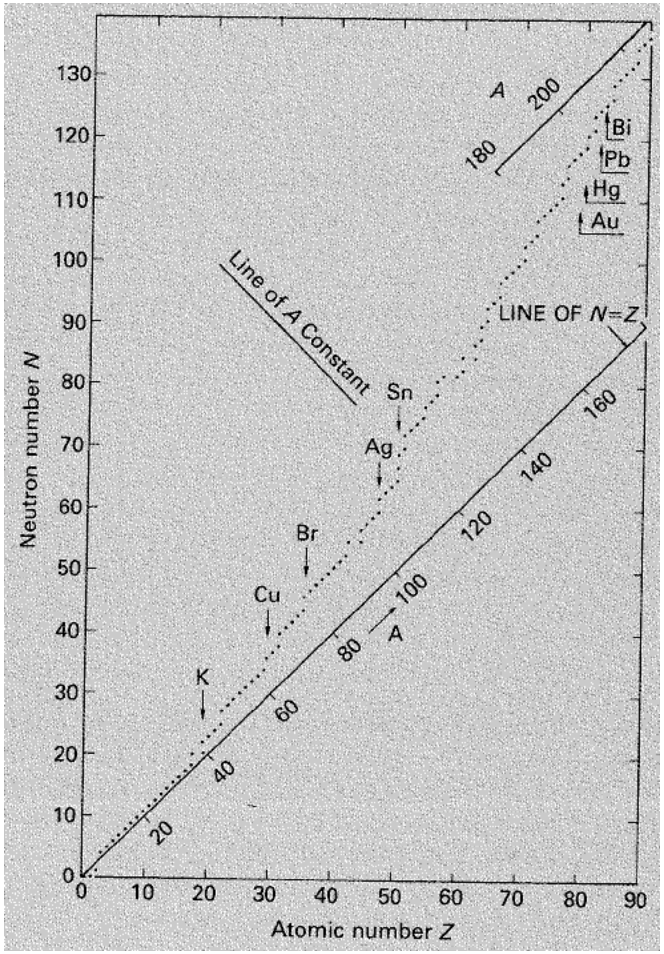
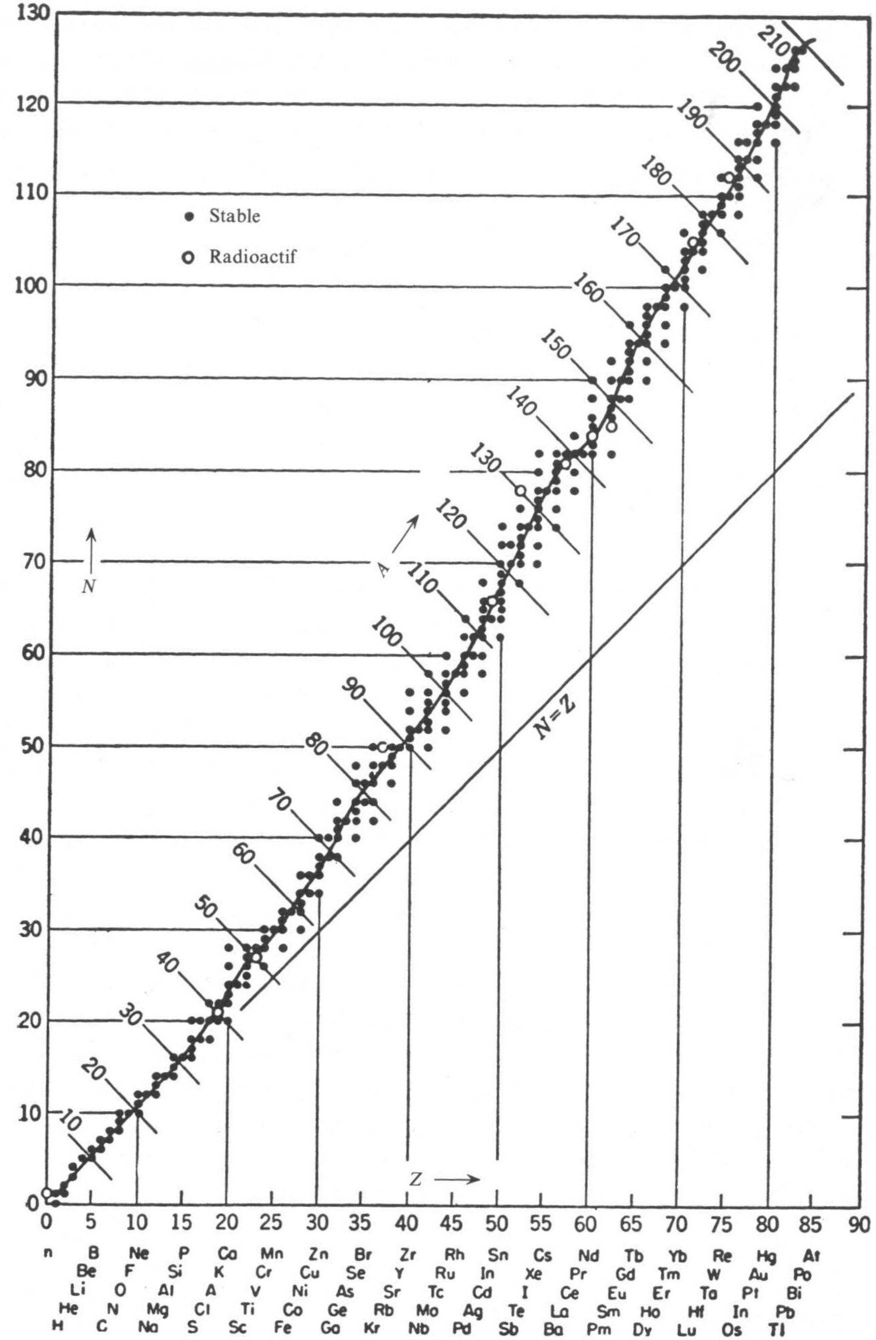


Chart of stable or long lived nuclei

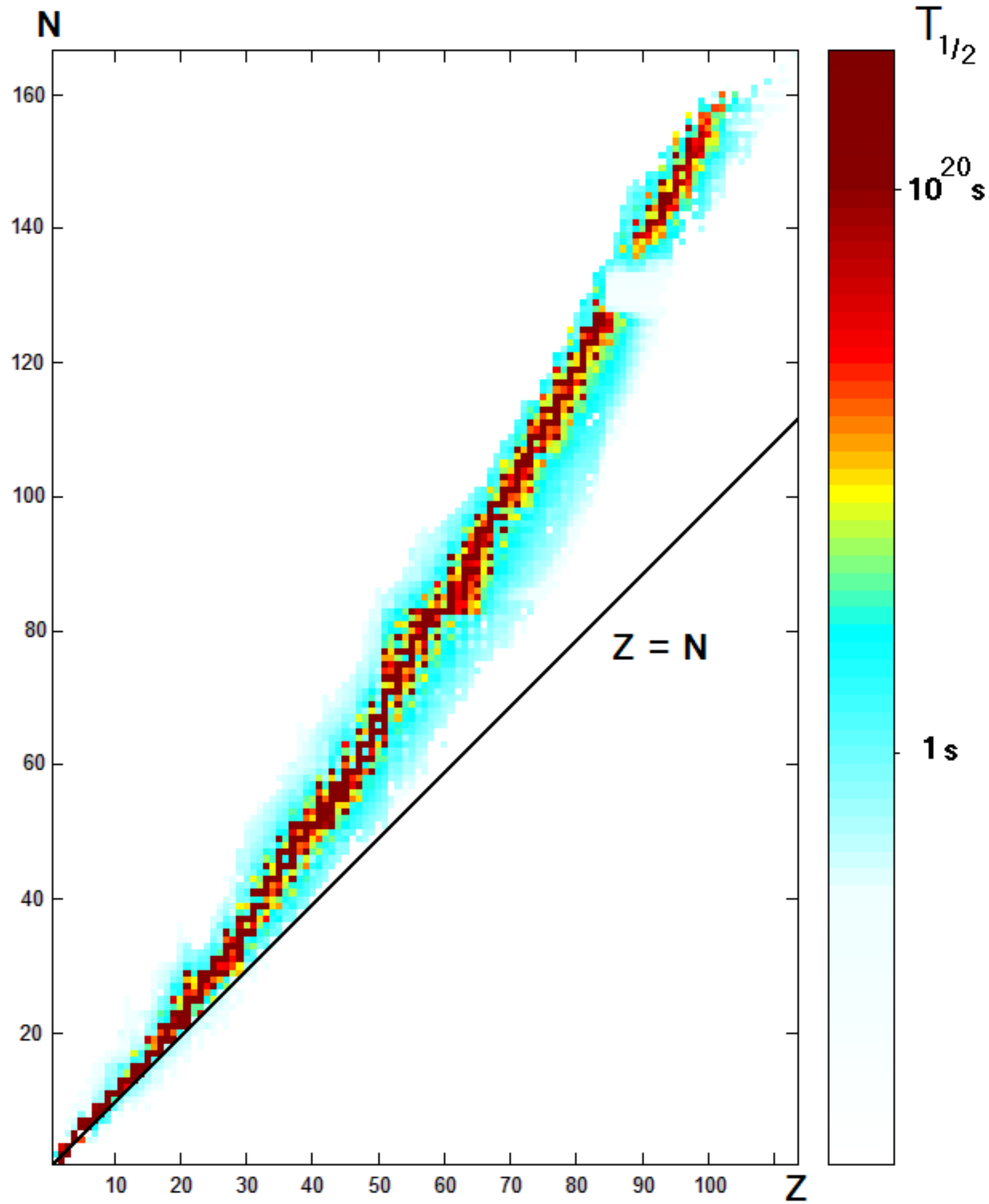


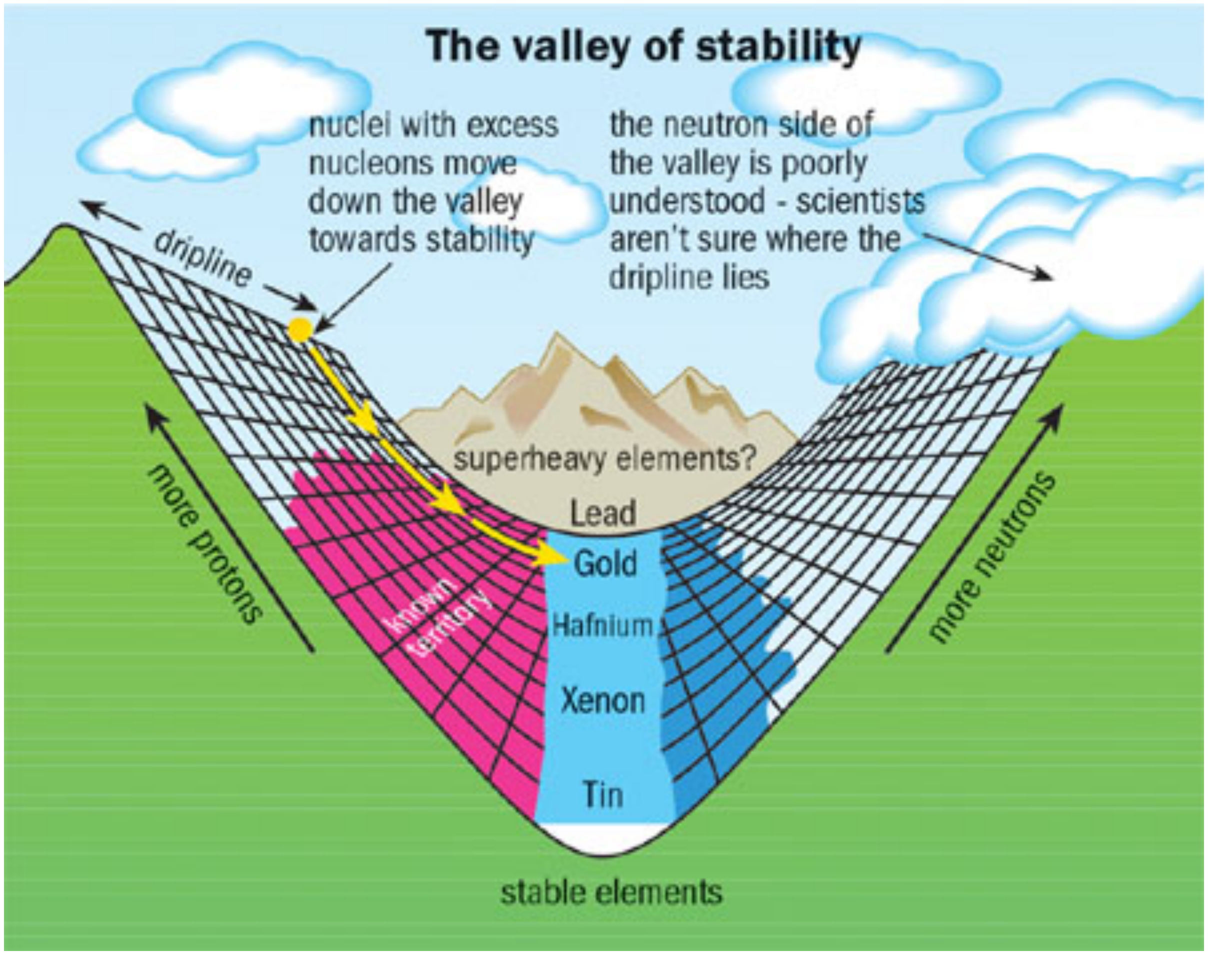
Line of stability:
line in the (N,Z) plan following the mean position of the stable nuclei for each A value

For even-A nuclei, there can be more than one stable nucleus.

• Bombarding the stable nuclei with nuclear projectile leads to the creation of artificial radio-isotope.
• The more these new nuclides are far from the line of stability, the shortest lifetime

➪ Valley of Stability




The valley of stability
nuclei with excess nucleons move down the valley towards stability
the neutron side of the valley is poorly understood - scientists aren't sure where the dripline lies
dripline
more protons
more neutrons
known territory
superheavy elements?
Lead
Gold
Hafnium
Xenon
Tin
stable elements

Key Observations

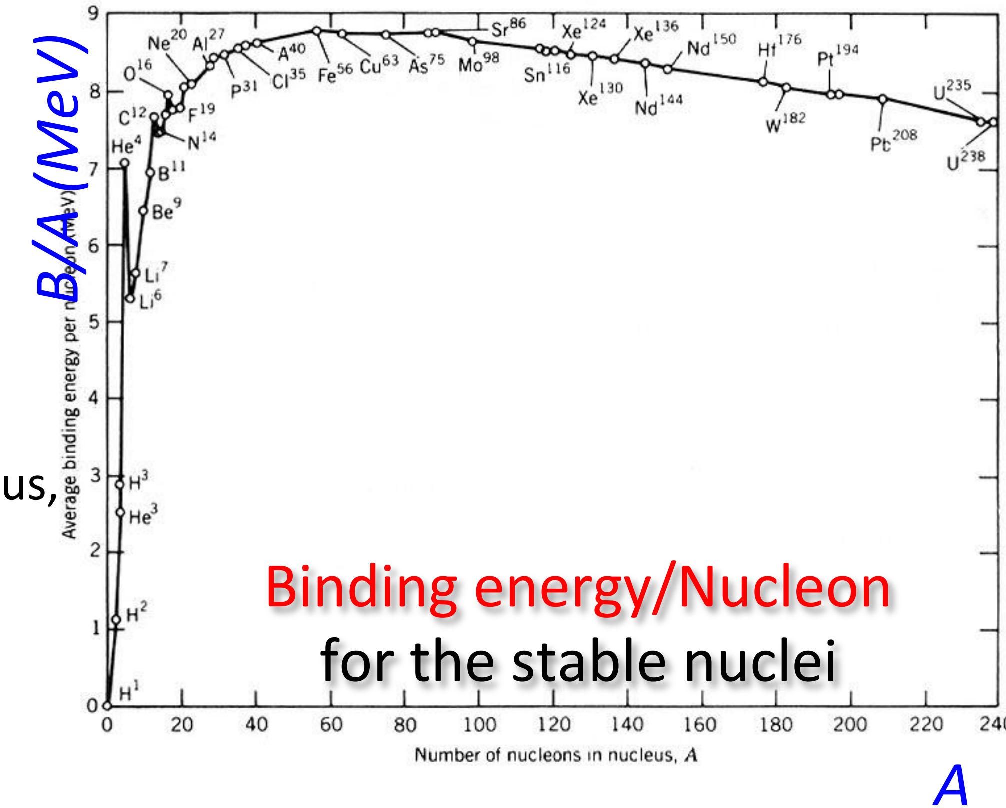
Stable nuclei are the nuclei which do not decay. Some long lived nuclei do not decay by the strong interaction, although they may transform by β and α emission (weak or electromagnetic) with long lifetime.

- 80 known elements have at least one stable element.
  From hydrogen (Z=1) to lead (Z=82) with the exception of Tc (Z=43) and Pr (Z=59). *(peculiar case of ^{209}Bi, Z=83)*
- 94 elements are found on earth, up to Plutonium (Z=94).
- There are 269 stables nuclei, and 339 isotopes found on earth
  (list on http://www.don-lindsay-archive.org/creation/isotope_list.html)
- 16 elements have a single isotope, on the other hand tin (Z=50) has 10 stable isotopes
- Amongst the stable nuclei:
  - 158 have both Z and N even (even-even nuclei)
  - 107 have either Z or N even (odd-A nuclei)
  - 4 have both Z and N odd (odd-odd nuclei)
    ^{2}H, ^{6}Li, ^{10}B, ^{14}N
    → Most odd-odd nuclei are highly unstable.
- There is only one stable isobar for each odd value of A
- Tend to have N=Z for light nuclei. Heavier nuclei have N>Z
  →N/Z ~ 1.5 for the heaviest stable nuclei
- Some values of Z and N exhibit larger number of isotopes and isotones.

1.2- Binding energies

Definition : $m({}^A_Z\mathrm{X})c^2 = Zm_pc^2 + Nm_nc^2 - B(A,Z)$

- ✓ B/A ≈ constant ≈ 8 MeV/nucleon for A>20 nuclei
  for comparison $B_e/A \leq 3keV$
- ✓ Broad maximum around A≈60 (Fe, Co, Ni), B/A≈8.7 MeV
- ✓ Light nuclei with A=4n (n integer) show peaks (α stability)



B/A ≈ constant – implies that in a nucleus, the nucleons are only attracted by nearby nucleons.

➪ Nuclear force is short range and saturated (saturated means each nucleus interacts with a limited number of neighbors; not with all nucleons)

1.3- Nuclear density

Using the Fermi parameterization:

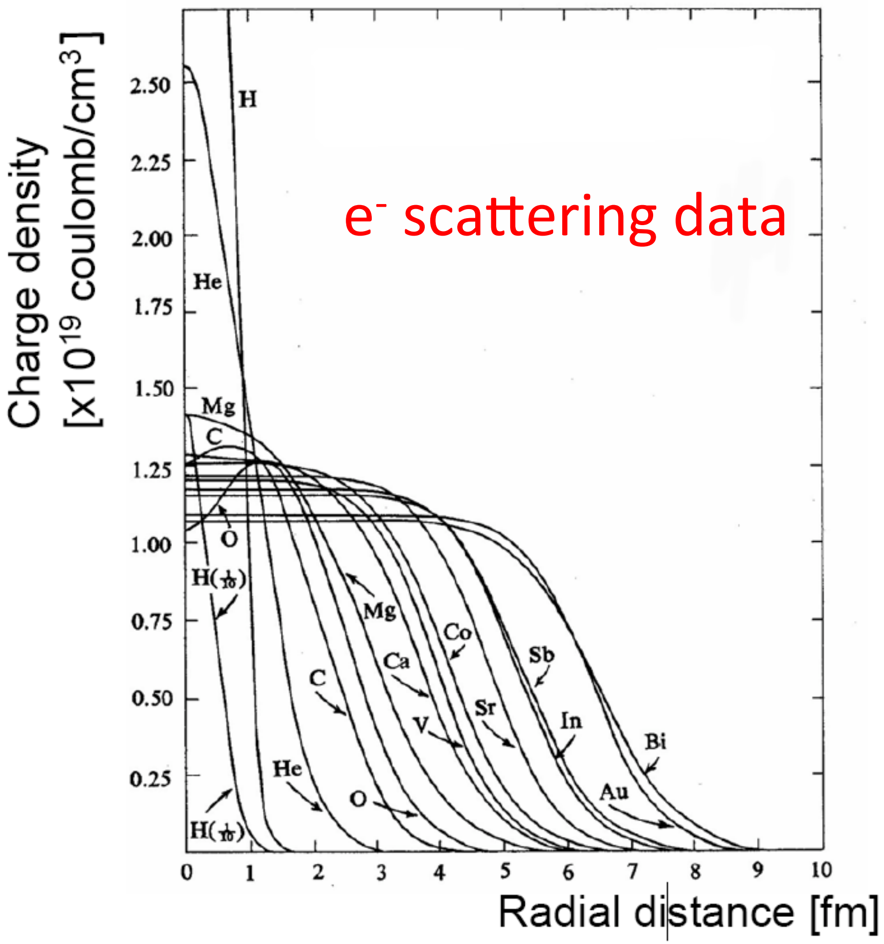
$$\rho(r) = \frac{\rho(0)}{1 + e^{\left(\frac{r-R}{d}\right)}}$$

Description of the nucleus as a sphere with a uniform interior density which drops to zero at the surface

For all nuclei

- $R \approx r_0 A^{1/3}$ with $r_0 = 1.2$ fm
- Central density almost identical for all nuclei.
  → again consistent with short-range saturated forces

➪ c.f. a droplet of liquid



2- Liquid Drop Model

Liquid drop

- Short range of the intermolecular forces
- Density independent of the drop size
- Heat required to evaporate fixed mass independent of the drop size

Nucleus

- Short range of the strong nuclear force
- Density independent of the nuclear size
- $B/A \approx$ constant

Basic hypothesis of the model:

- The nuclear force has a short range (~1 fm) and can be assimilated as a contact interaction
- Because of a short range repulsion (hard core component of the nuclear interaction), the nucleus is made of incompressible matter.
  → Drop of incompressible nuclear fluid

Other assumption:

- Spherical nucleus

Semi-Empiric Mass Formula

(Sometimes also called Weizsäcker's formula)

- Formula used to approximate the mass and various other properties of an atomic nucleus. As the name suggests, it is partially based on theory and partly on empirical measurements.
- The theory is based on the liquid drop model, and can account for most of the terms in the formula, and gives a rough estimates for the values of the coefficients. It was first formulated in 1935 by German physicist Carl von Weizsäcker, and although refinements have been made to the coefficients over the years, the form of the formula remains the same today.

$$B(A,Z) = \underbrace{u_v A - u_s A^{2/3} - u_c \frac{Z^2}{A^{1/3}}}_{\text{Liquid drop part}} \underbrace{- u_T \frac{(N-Z)^2}{A} + \delta}_{\text{Nuclear specific terms}}$$

- U_v, u_s, u_c, u_T and δ are parameters in principle independent of A and Z, whose numerical value should be adjusted to accurately reproduce the experimental nuclear masses.

2.1- Volume term

$$B_{\text{vol}} = u_v A$$

- The basis form this term is the strong nuclear force affecting both neutrons and protons.
- Increases B and reduces mass by a constant amount per nucleon.
- The volume term is proportional to the number of nucleons:
  - Charge independence of the strong nuclear force.
  - Short range and saturated force.
  - B/A ≈ Constant instead of B/A ≈ A for long range interactions.
- The value of the u_v parameter can be estimated in the Fermi gaz model (nucleus treated as a Fermi ball of A nucleon): u_v = 17 MeV
- The best fitted value (least squares fit to experimental values) is u_v = 15.8 MeV

2.2 Surface term

$$B_{\text{surf}} = -u_s A^{2/3}$$

- This term is also based on the strong nuclear force, and is a correction to the volume term and reduces B.
- The volume term suggest that each nucleon interacts with a constant number of nucleons, independent of A. While this is very nearly true for nucleons deep within the nucleus, the nucleons on the surface of the nucleus have fewer nearest neighbors, justifying this correction. This can also be thought of as a surface tension term, and indeed a similar mechanism creates surface tension in liquids.
- If the volume of the nucleus is proportional to A, then the radius should be proportional to $A^{1/3}$ and the surface area to $A^{2/3}$
  $\rightarrow B_{surf} \propto A^{2/3}$.
- u_s should have a similar order of magnitude as u_v.
- The best fitted value (least squares fit to experimental values) is
  u_s = 18.3 MeV

2.3 Coulomb term

$$B_{\text{coul}} = -u_c \frac{Z^2}{A^{1/3}}$$

- Coulomb or electrostatic term reducing B.
- The basis for this term is the electrostatic repulsion between protons. To a very rough approximation, the nucleus can be considered a sphere of uniform charge density. The potential energy of such a charge distribution (radius R) can be shown to be :

$$E_c = \frac{3}{5}\frac{Z^2\alpha\hbar c}{R}$$

- Since $R=r_0A^{1/3}$, the value of the coulomb term is:

$B_{\text{coul}} = -\frac{3\alpha\hbar c}{5r_0}\frac{Z^2}{A^{1/3}}$ hence $u_c = \frac{3\alpha\hbar c}{5r_0} = 0.72\ \text{MeV}$

- Rigorously, the proton charge does not interact with itself
→ replace Z^2 with Z(Z-1) because of the discrete nuclear charge. Very small effect
- The best fitted value (least squares fit to experimental values) is
u_c = 0.714 MeV

At this point, we have implemented a basic liquid drop model:

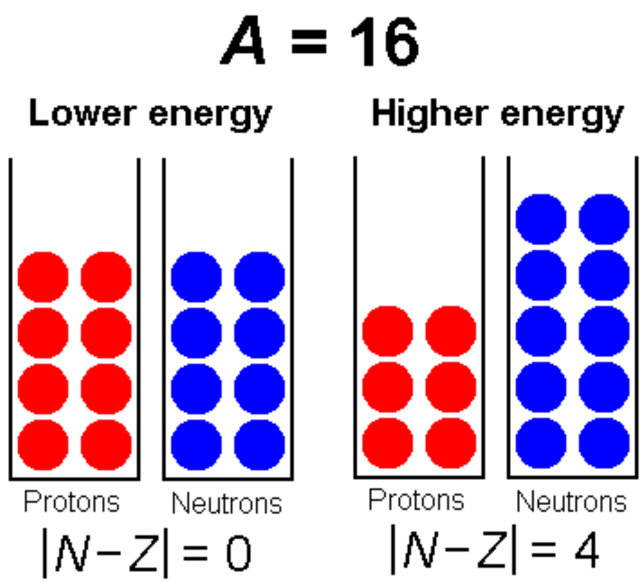
$$B(A,Z) = u_v A - u_s A^{2/3} - u_c \frac{Z^2}{A^{1/3}}$$

It does not account for two of the experimental observations:

1. N≈Z
2. Nucleons tend to pair

2.4- Asymmetry term

- Due to the Pauli exclusion principle
- Since p and n are confined in the nuclear volume, their energies are discrete
- 2 fermions can not occupy exactly the same quantum state → as more particles are "added", these particles must occupy higher energy levels, increasing the total energy of the nucleus (and decreasing the binding energy).
- Protons and neutrons occupy different quantum states. If there are significantly more neutrons than protons in a nucleus, some of the neutrons will be higher in energy than the available states for the protons.
  → Moving some particles from the neutron side to the proton side (in other words changing some neutrons into protons) would significantly decrease the energy. The imbalance between the number of protons and neutrons causes the energy to be higher than it needs to be, for a given number of nucleons. This is the basis for the asymmetry term.



$$B_{\mathrm{asym}} = -u_T \frac{(N-Z)^2}{a}$$

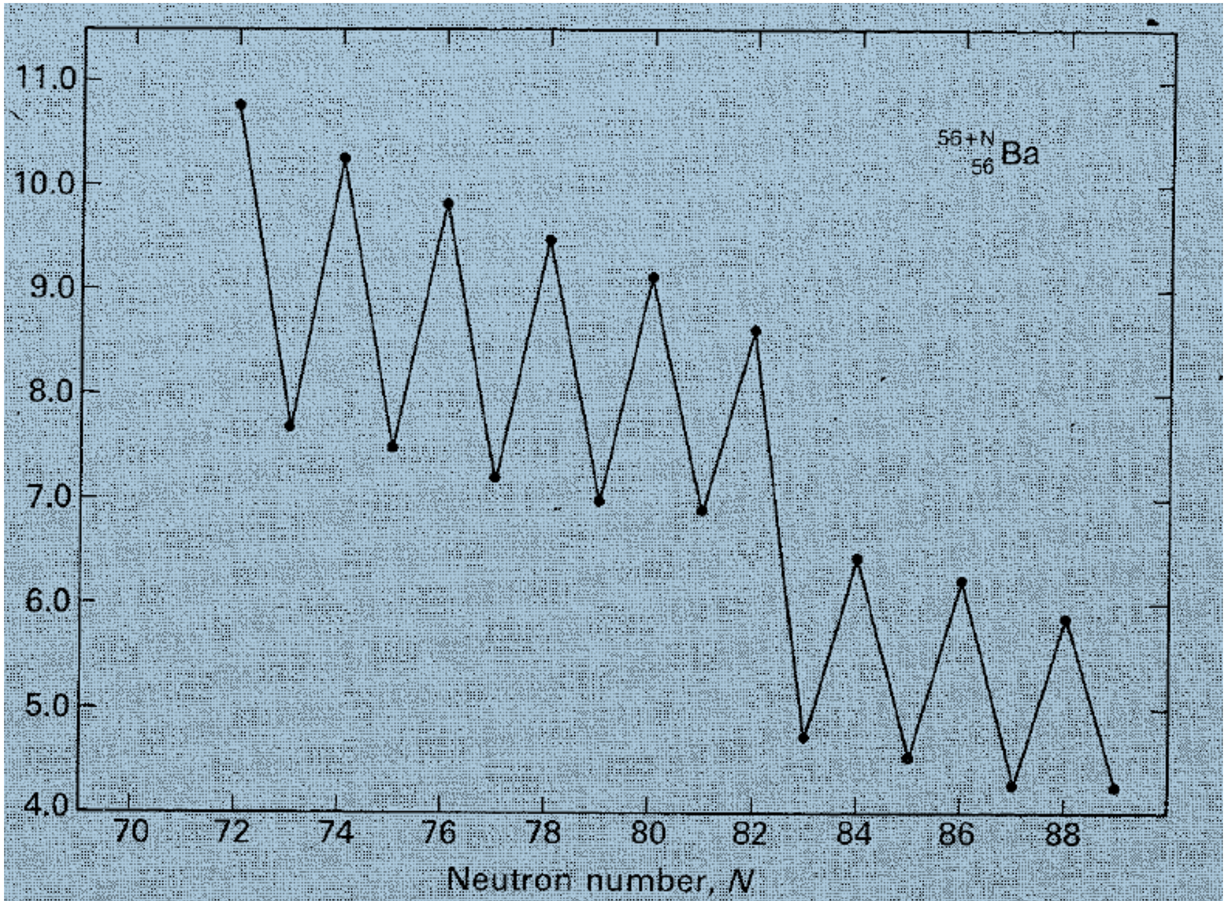
- This effect is not based on any of the fundamental forces (gravitational, electromagnetic, etc.), only the Pauli exclusion principle.
- Asymmetry term can be derived from Fermi gas model (see next chapter):
  - the form of the asymmetry term is $(N\text{-}Z)^2/A$ or $(A\text{-}2Z)^2/A$
  - The value of the u_T parameter is $E_F/3$
- The best fitted value (least squares fit to experimental values) is
  u_T = 23.2 MeV

2.5- Pairing term

$$B_{\text{pair}} = \begin{cases} +\delta & \text{even-even nuclei} \\ 0 & \text{odd-A nuclei} \\ -\delta & \text{odd-odd nuclei} \end{cases} \qquad \text{with } \delta = \delta(A) = u_p\, A^{-1/2}$$

- Pairing interaction energetically favors the formation of pairs of like nucleons (pp, nn) with spins ↑↓and symmetric space wavefunction.
- This term captures the effect of spin-coupling
- Form is simply empirical
- The best fitted value (least squares fit to experimental values) is
  u_p = 12 MeV

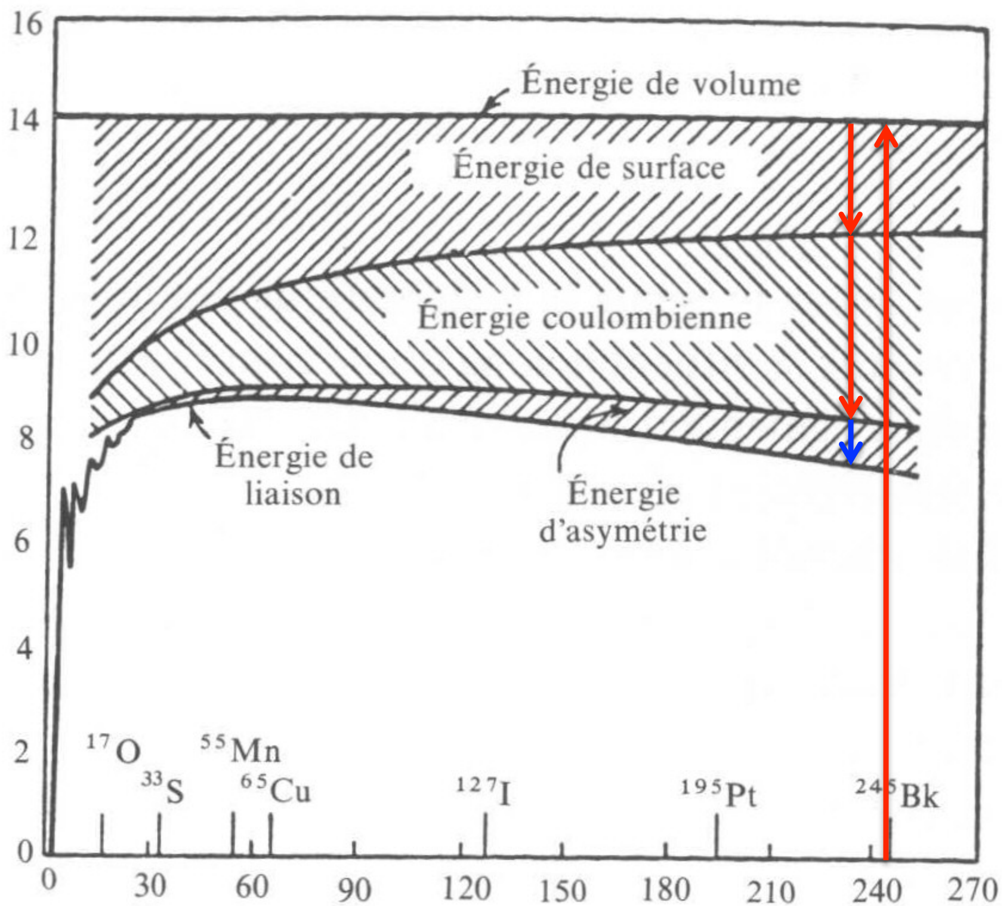
Separation energy of the last neutron in the isotopes of $_{56}$Ba



.4 The energy required to remove the last neutron (the **neutron separation**
) from the isotopes of barium ($_{56}$Ba). This energy is about 2 MeV greater when
en than when N is odd. This 2 MeV represents the energy required to break the
; favoured for like nucleons. The irregularity at $N=82$ is one of the effects that

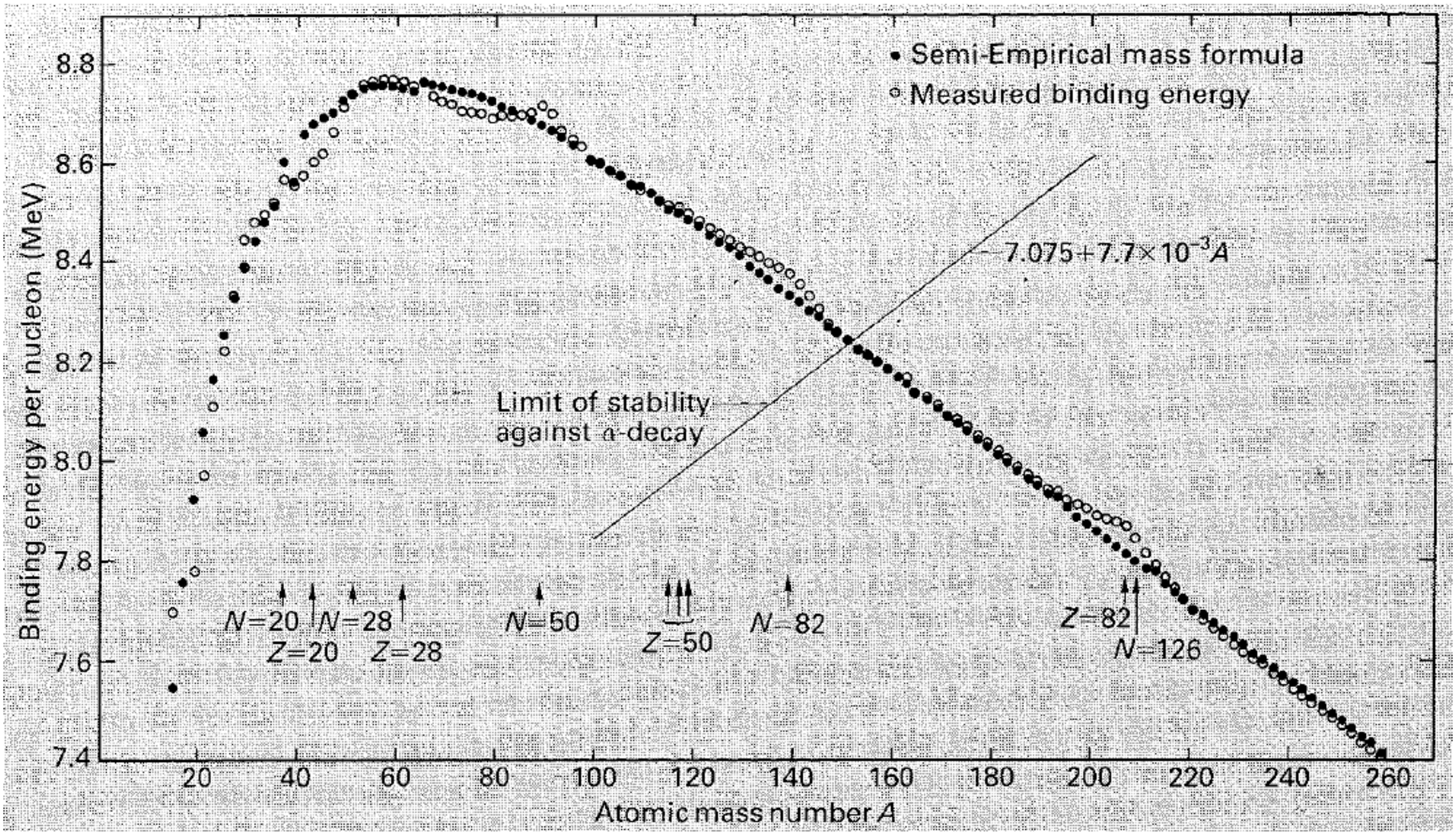
The semi-empiric mass formula

$$B(A,Z) = u_v A - u_s A^{2/3} - u_c \frac{Z^2}{A^{1/3}} - u_T \frac{(N-Z)^2}{A} + \delta$$



- This formula works Well for the nuclei with A>20.
- For the very light nuclei, the agreement is poor :
  - ✓ To few nucleons to allow a description in term of a drop a nuclear matter
  - ✓ predominance of shell effects

Binding energy per nucleon as a function of A for the odd-A nuclei from A=15-259



For A>20, the precision is for the majority of cases better than 0.1 MeV

On the value of the parameters:

- As they are determined from experimental data, they can have different values depending on the method used for the extraction.
- Example of 3 set of parameters (given in MeV).

	Least square fit	Wapstra	Rohlf
u_v	15.8	14.1	15.75
u_s	18.3	13	17.8
u_c	0.714	0.595	0.711
u_T	23.2	19	23.7
u_p	12	✗	✗
$\delta_{even\text{-}even}$	✗	33.5	11.18
$\delta_{odd\text{-}odd}$	✗	-33.5	-11.18
$\delta_{even\text{-}odd}$	✗	0	0

- Wapstra: Atomic Masses of Nuclides, A. H. Wapstra, Springer, 1958
- Rohlf: Modern Physics from a to Z0, James William Rohlf, Wiley, 1994

3- Applications

- Valley of stability
- β stability
- α stability
- Fission
- Neutron stars

$$\begin{aligned} M(A,Z) &= ZM(^1H) + (A-Z)m_n - u_v A + u_s A^{2/3} \\ &+ u_c \frac{Z^2}{A^{1/3}} + u_T \frac{(A-2Z)^2}{A} - \delta(A) \end{aligned}$$

3.1- Stability Line

- Let's focus on a series of odd-A nuclei (no pairing term)

$$B(A,Z) = \alpha Z^2 + \beta Z + \gamma$$

$$\text{with} \begin{cases} \alpha = -\left(\frac{u_c}{A^{1/3}} + \frac{4u_T}{A}\right) \\ \beta = 4u_T \\ \gamma = (u_v - u_T)A - u_s A^{2/3} \end{cases}$$

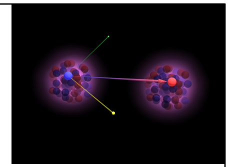
- B(A,Z) is maximum at $Z=Z_0$ → $\left.\frac{\partial B}{\partial Z}\right|_{A=\text{const}} = 0$

$$Z_0 = \frac{-\beta}{2\alpha} = \frac{2u_T}{\frac{u_c}{A^{1/3}} + \frac{4u_T}{A}} = \frac{2A}{\frac{u_c}{u_T}A^{2/3} + 4}$$

- Examples :
  A=238, Z_0=91.8 (U: Z=92) ; A=55, Z_0=24.8 (Mn: Z=25)

3.2 β stability

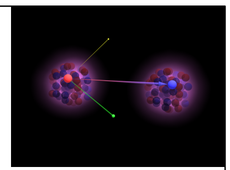
- β⁻ decay



$${}^{A}_{Z}\mathrm{X} \longrightarrow {}^{A}_{Z+1}\mathrm{Y} + e^{-} + \bar{\nu}_e$$

$$Q_{\beta-} = M(X) - M(Y)$$

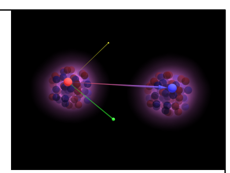
- β⁺ decay



$${}^{A}_{Z}\mathrm{X} \longrightarrow {}^{A}_{Z-1}\mathrm{Y} + e^{+} + \nu_e$$

$$Q_{\beta+} = M(X) - M(Y) - 2m_e$$

- Electron capture



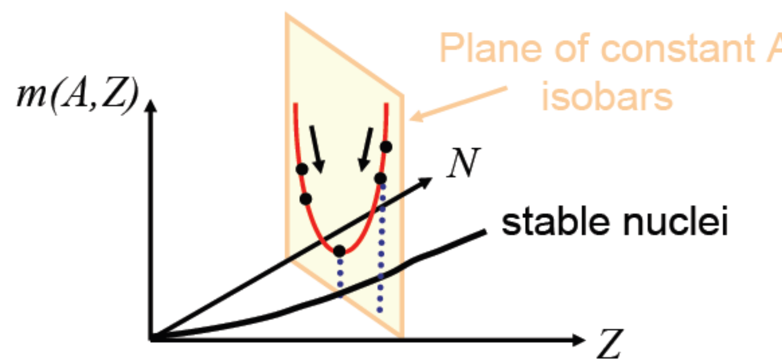
$$e^{-} + {}^{A}_{Z}\mathrm{X} \longrightarrow {}^{A}_{Z-1}\mathrm{Y} + \nu_e$$

$$Q_{ec} \simeq M(X) - M(Y)$$

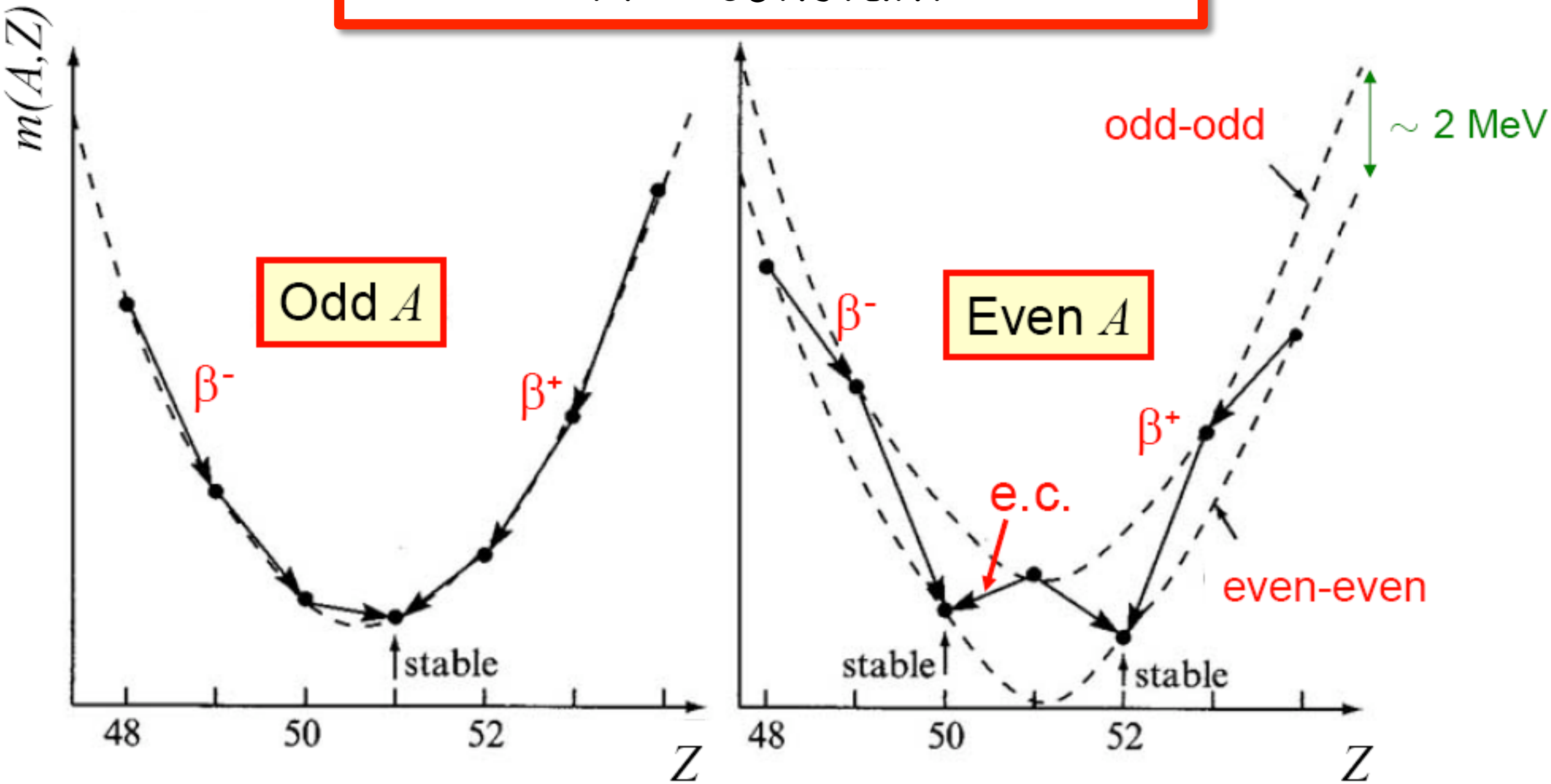
$$\begin{aligned} M(A,Z) \quad = \quad & Z M({}^{1}H) + (A-Z)m_n - u_v A \\ & + u_s A^{2/3} + u_c \frac{Z^2}{A^{1/3}} \\ & + u_T \frac{(A-2Z)^2}{A} - \delta(A) \end{aligned}$$

β decay:

- Isobaric (A constant)
- Z changes by ±1
- M(A,Z) is quadratic in Z (min for Z_0)
  → mass parabola

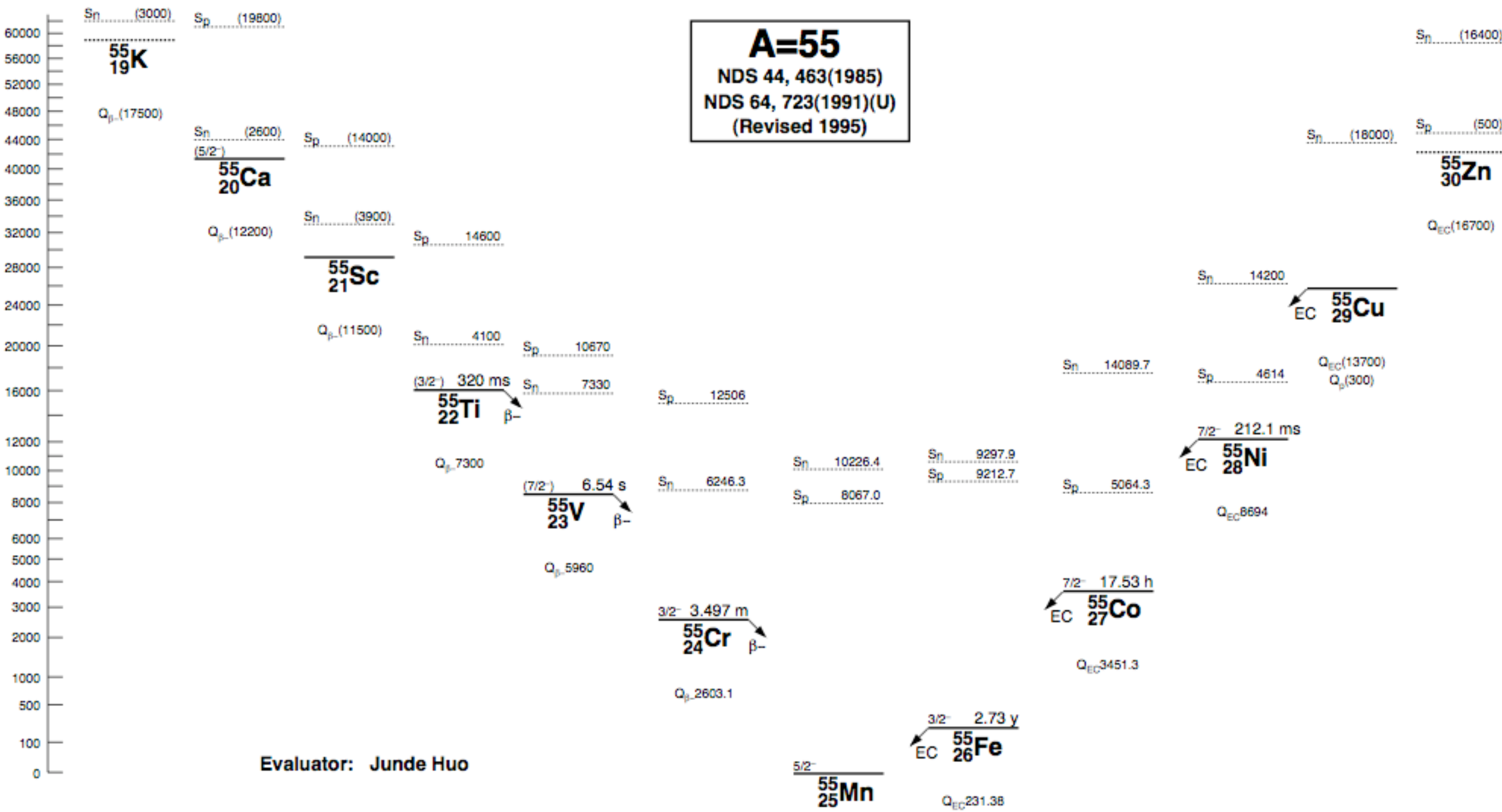


Examples of isobaric chains
A = constant

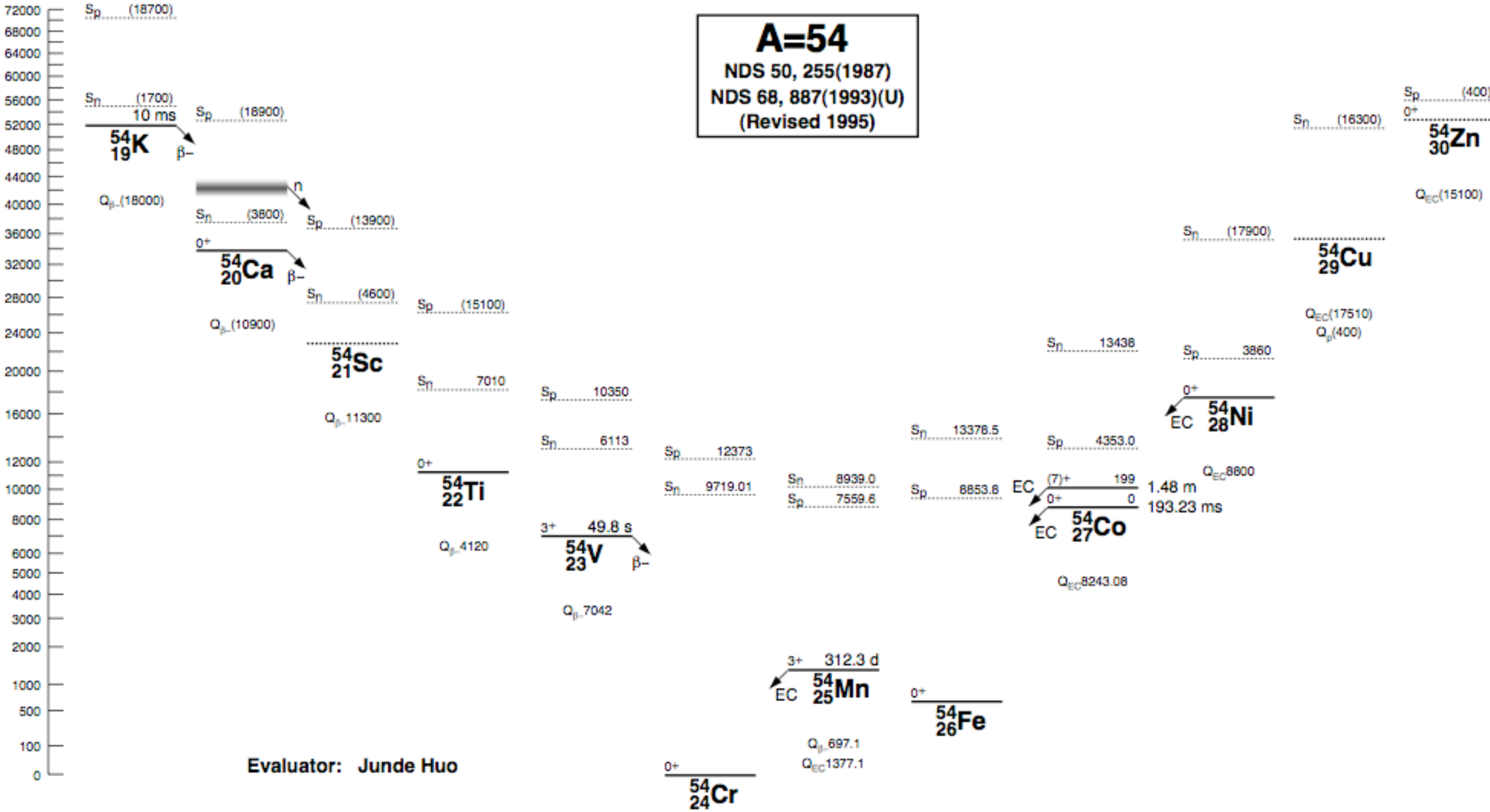


Usually only one isotope stable against β-decay; maybe two.

Typically two even-even nuclides stable against β-decay; no odd-odd ones.



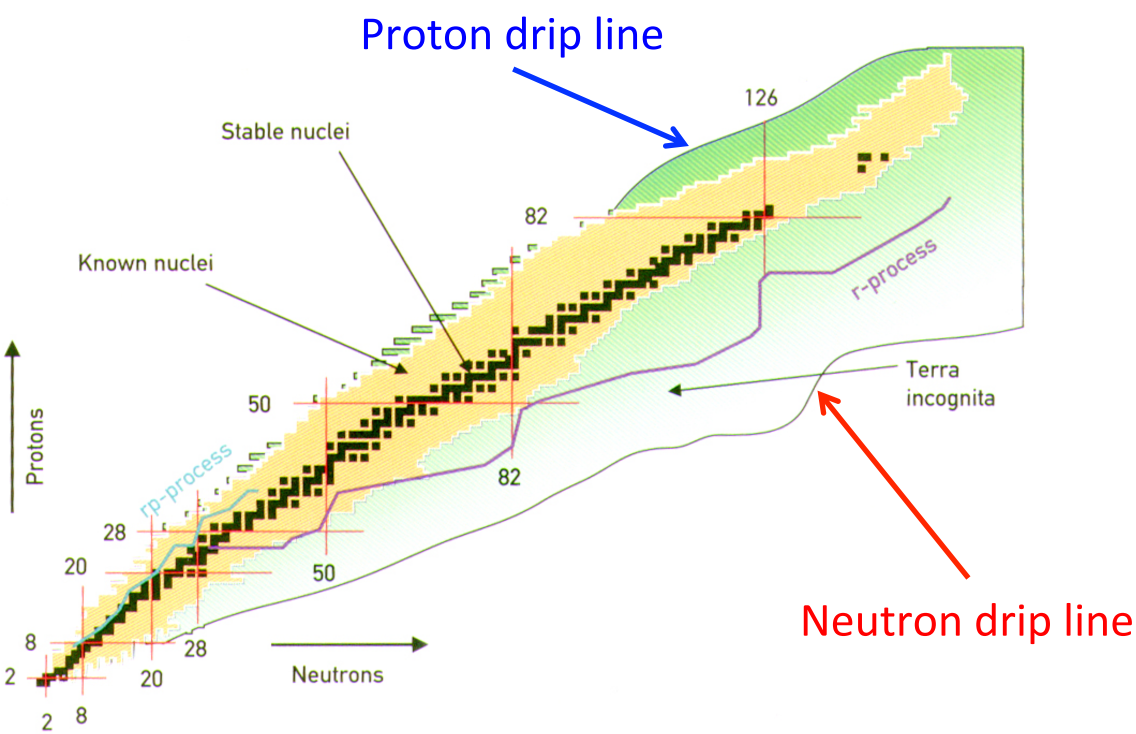
A=55
NDS 44, 463(1985)
NDS 64, 723(1991)(U)
(Revised 1995)
Evaluator: Junde Huo
Sn (3000)
55 19 K
Qβ- (17500)
Sp (19800)
Sn (2600)
(5/2-)
55 20 Ca
Qβ- (12200)
Sp (14000)
Sn (3900)
55 21 Sc
Qβ- (11500)
Sp 14600
Sn 4100
(3/2-) 320 ms
55 22 Ti
β-
Qβ- 7300
Sp 10670
Sn 7330
(7/2-) 6.54 s
55 23 V
β-
Qβ- 5960
Sp 12506
Sn 6246.3
3/2- 3.497 m
55 24 Cr
β-
Qβ- 2603.1
Sn 10226.4
Sp 8067.0
5/2-
55 25 Mn
Sn 9297.9
Sp 9212.7
3/2- 2.73 y
EC
55 26 Fe
QEC 231.38
Sn 14089.7
Sp 5064.3
7/2- 17.53 h
EC
55 27 Co
QEC 3451.3
Sn 14200
Sp 4614
7/2- 212.1 ms
EC
55 28 Ni
QEC 8694
Sn (18000)
EC
55 29 Cu
QEC (13700)
Qp (300)
Sn (16400)
Sp (500)
55 30 Zn
QEC (16700)
60000
56000
52000
48000
44000
40000
36000
32000
28000
24000
20000
16000
12000
10000
8000
6000
5000
4000
3000
2000
1000
500
100
0


A=54
NDS 50, 255(1987)
NDS 68, 887(1993)(U)
(Revised 1995)
Evaluator: Junde Huo
54 19 K 10 ms β−
Qβ-(18000)
54 20 Ca 0+ β−
Qβ-(10900)
54 21 Sc
Qβ-11300
54 22 Ti 0+
Qβ-4120
54 23 V 3+ 49.8 s β−
Qβ-7042
54 24 Cr 0+
54 25 Mn 3+ 312.3 d EC
Qβ-697.1
QEC1377.1
54 26 Fe 0+
54 27 Co (7)+ 199 1.48 m 0+ 0 193.23 ms EC
QEC8243.08
54 28 Ni 0+ EC
QEC8800
54 29 Cu
QEC(17510)
Qp(400)
54 30 Zn 0+
QEC(15100)

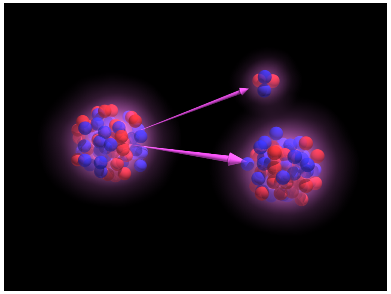
- For each A, it is possible to calculate the maximum value of Z (resp. N), where the proton (resp. neutron) separation energy is zero.

$$S_n = B(A, Z) - B(A-1, Z)$$
$$S_p = B(A, Z) - B(A-1, Z-1)$$

- These lines are called the proton drip line and the neutron drip line. They represents the boundary for proton-rich and neutron-rich nuclei existence



3.3 α stability



- α decay: ${}^{A}_{Z}\mathrm{X} \longrightarrow {}^{A-4}_{Z-2}\,\mathrm{Y} + {}^{4}_{2}\,\mathrm{He}$

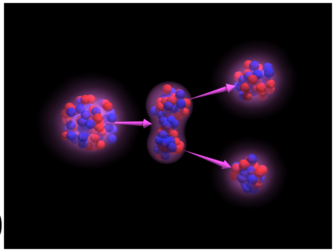
$$\begin{aligned} Q_\alpha &= M(X) - M(Y) - M(^4He) \\ &= M(A,Z) - M(A-4,Z-2) - M(4,2) \\ &= -B(A,Z) + B(A-4,Z-2) + B(4,2) \end{aligned}$$

$$Q_\alpha = B_\alpha - 2\frac{\partial B}{\partial Z} - 4\frac{\partial B}{\partial A}$$

$$Q_\alpha = 28.3\ \mathrm{MeV}\ - 4u_v + \frac{8}{3}\frac{u_s}{A^{1/3}} + 4u_c\frac{Z(3A-Z)}{3A^{4/3}} - 4u_T(1-\frac{2Z}{A})^2$$

- For $Z \approx A/2$, $Q_\alpha > 0$ for $A > 100$
- For $Z \approx Z_0$, $Q_\alpha > 0$ for $A > 150$
- For all these nuclei (even along the stability valley), α decay is energetically possible.
- In fact Q_α is modulated by the shell effects

3.4 fission stability



Fission: $^{A}_{Z}\mathrm{X} \longrightarrow ^{A_1}_{Z_1}\mathrm{Y} + ^{A_2}_{Z_2}\mathrm{W}$

- Expect spontaneous fission to occur if

$$Q_f = M(X) - M(Y) - M(W) = B(A_1, A_1) + B(A_2, Z_2) - B(A, Z) > 0$$

- Lets define: $y = \frac{A_1}{A} = \frac{Z_1}{Z}$ and $1 - y = \frac{A_2}{A} = \frac{Z_2}{Z}$
- Then:

$$Q_f = u_s A^{2/3}\left(1 - y^{2/3} - (1-y)^{2/3}\right) + u_c \frac{Z^2}{A^{1/3}}\left(1 - y^{5/3} - (1-y)^{5/3}\right)$$

- Maximum energy released when $\frac{\partial Q_f}{\partial y} = 0$

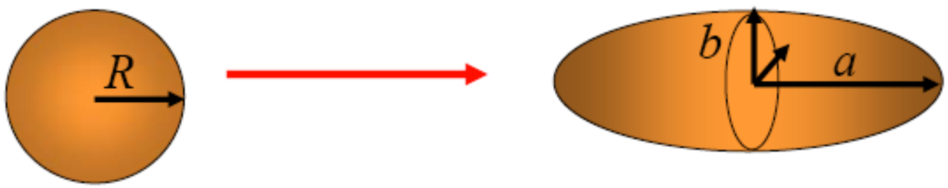
$$\frac{\partial Q_f}{\partial y} = \frac{2}{3}u_s A^{2/3}\left(-y^{-1/3} + (1-y)^{2/3}\right) + \frac{5}{3}u_c \frac{Z^2}{A^{1/3}}\left(-y^{2/3} + (1-y)^{2/3}\right) = 0$$

  occurs when y=1/2

- For symmetric fission (y=1/2), the energy released is $Q_f^{\mathrm{sym}} = 0.37 u_c \frac{Z^2}{A^{1/3}} - 0.26 u_s A^{2/3}$
- Example: for ^{238}U, $Q_f \approx 200$ MeV ~ 10^6 × energy released in chemical reaction

3.5 Nucleus deformation

- Investigate the simplest deformation of the nucleus: ellipsoid shape



$$a = R(1+\varepsilon) \qquad \varepsilon << 1$$
$$b = R(1+\varepsilon)^{-1/2}$$

- The volume term is unchanged (incompressible matter): $V = \frac{4}{3}\pi R^3 \Rightarrow \frac{4}{3}\pi ab^2$
- Change in surface term: $B_{\text{surf}} = -u_s A^{2/3} \Rightarrow -u_s A^{2/3}(1 + \frac{2}{5}\varepsilon^2))$
- Change in coulomb term: $B_{\text{coul}} = -u_c \frac{Z^2}{A^{1/3}} \Rightarrow -u_c \frac{Z^2}{A^{1/3}}(1 - \frac{1}{5}\varepsilon^2)$
- No changes if the asymmetry an pairing term.
- Binding energy of a deformed nucleus: $B_{\text{def}} = B_{\text{sph}} - \frac{2}{5}u_s\varepsilon^2 A^{2/3} + \frac{1}{5}u_c\varepsilon^2 \frac{Z^2}{A^{1/3}}$
- A deformed shape is more stable than a spherical one if:

$$B_{\text{def}} > B_{\text{sph}} \Longrightarrow \frac{Z^2}{A} > \frac{2a_s}{a_c} \simeq 47$$

- cf fission criteria

3.6- Neutron star

- Let's describe a neutron star (mass M, radius R), as a BIG nucleus containing only neutrons (A neutrons)
  ➪ $M = A\ m_n$ and $R = r_0\ A^{1/3}$
- The self gravitational energy is: $E_G = -\frac{3}{5}\frac{\mathcal{G}M^2}{R}$
- The gravitational binding energy is:

$$B_G = +u_G A^{5/3} \quad \text{with} \quad u_G = \frac{3}{5}\frac{\mathcal{G}m_n^2}{r_0} = 5.74\ 10^{-37}\ \text{MeV}$$

  negligible for A ~ 100
- For larger value of A, this term become sizable and the surface term is negligible versus the volume term. Asking for the overall stability of the star ($B_{nucl}+B_{grav}>0$) gives:

$$A_{\min} > \left(\frac{u_T - u_v}{u_G}\right)^{3/2} = 4.9\ 10^{55}$$

Corresponding to:

$$\begin{cases} M = A_{\min} \times m_n = 8.2\ 10^{28}\,\text{kg} = 0.04\ M_{\text{soleil}} \\ R = r_0 A_{\min}^{1/3} = 4.4\ \text{km} = 6\,10^{-6}\ R_{\text{soleil}} \end{cases}$$