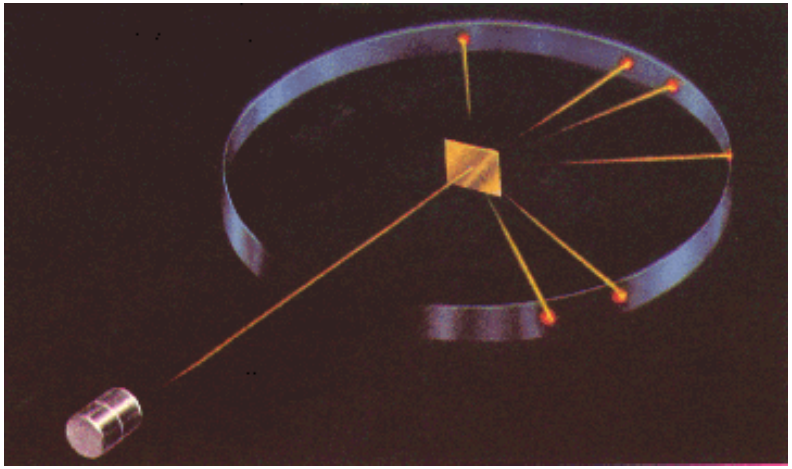
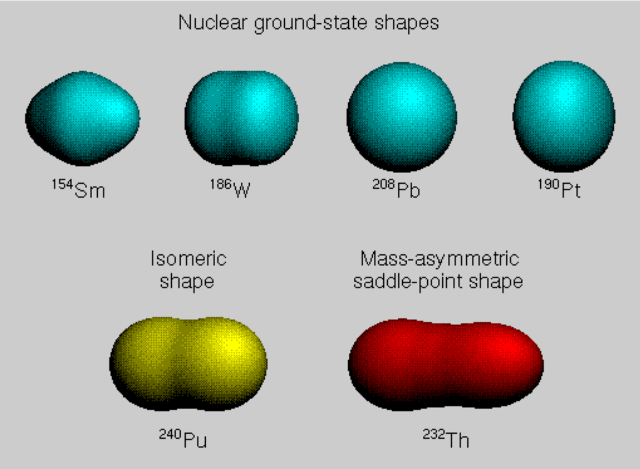
Chapter 3
The size and shape of the nuclei





Outline/Plan

1- Introduction

The size of nucleus is determined using various methods

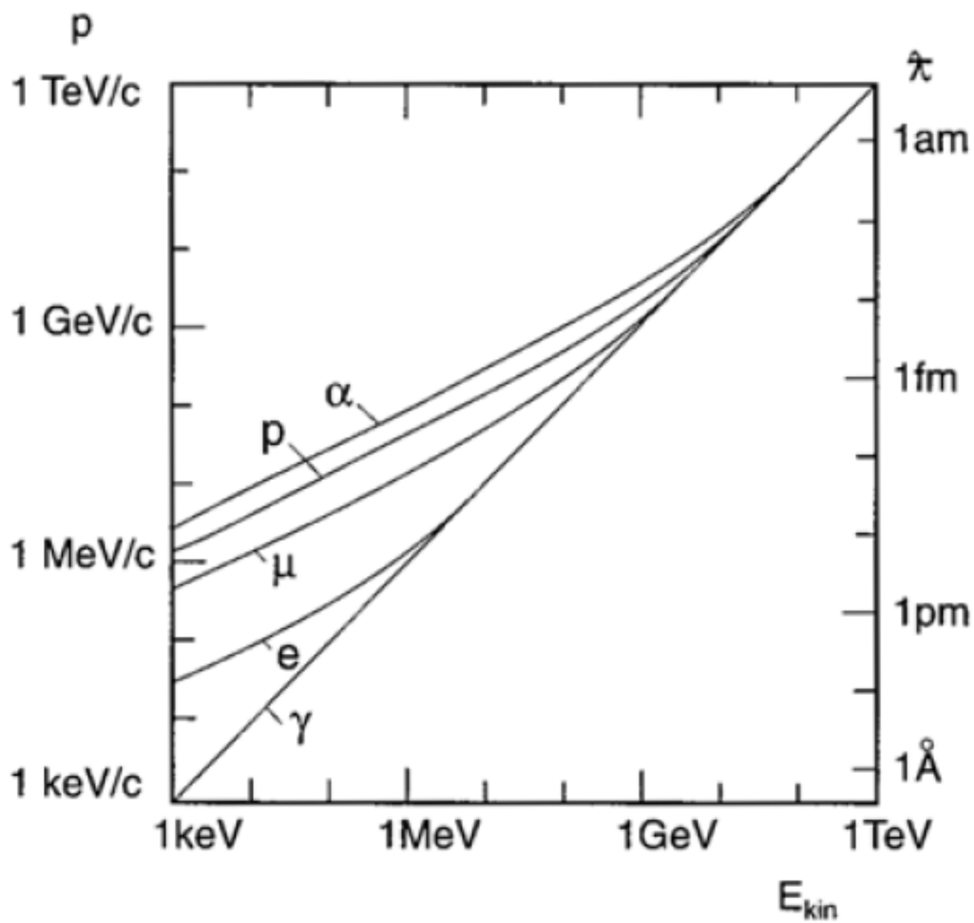
- Methods using electromagnetic interaction
  - Electron scattering
  - X-ray spectrum of muonic atom
  - Mirror nuclei
- Methods using the strong nuclear interaction
  - α scattering
  - n scattering
  - p, π scattering
  - Lifetimes of α particle emitters

The most important of these methods are based on angular scattering of incoming particles

An incoming particle can probe the matter up to a distance:

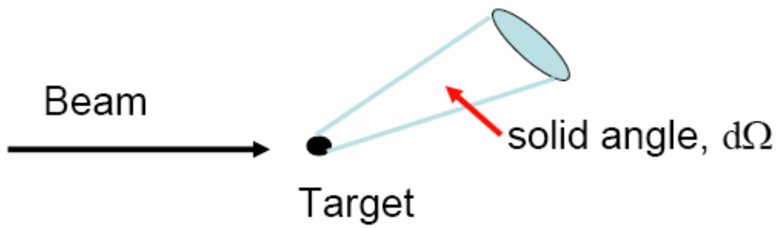
$$\lambda\!\!\!^{-} = \frac{\hbar}{p} = \frac{\hbar c}{\sqrt{K(K+2mc^2)}}$$

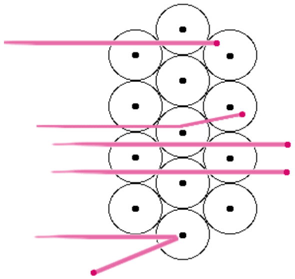
With K: kinetic energy



2- Differential cross-section

The angular distribution of scattered particles is not necessarily uniform.





$\Delta N_{d\Omega}$ = Number of particles scattered into dΩ

$$\Delta N_{\mathrm{d}\Omega} = \Phi N_T \mathrm{d}\sigma$$

$$\boxed{\frac{\mathrm{d}\sigma}{\mathrm{d}\Omega} = \frac{\Delta N_{\mathrm{d}\Omega}}{\Phi\, N_T\, \mathrm{d}\Omega}}$$

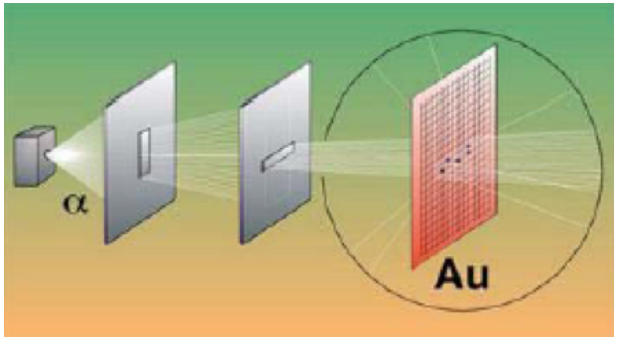
ANGULAR DIFFERENTIAL CROSS-SECTION
units : L^2/Sr

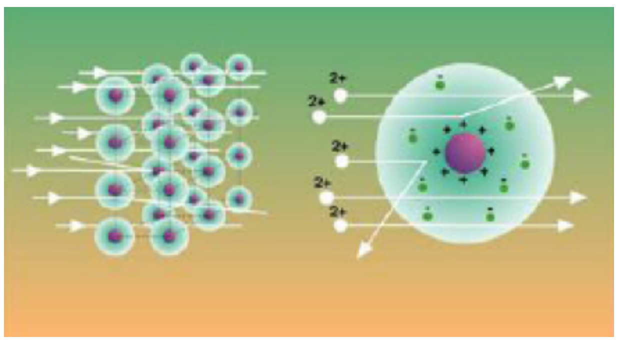
This differential cross section is the number of particles scattered per unit time and solid angle divided by the incident flux and by the number of target nuclei defined by the beam area

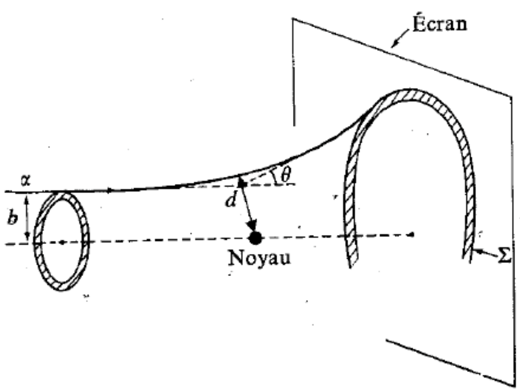
- Most experiment do not cover 4π solid angle
- Angular distributions provide more information about the mechanism of the interaction
- Integration on the whole solid angle gives the total scattering cross-section:

$$\sigma_s = \int \frac{\mathrm{d}\sigma}{\mathrm{d}\Omega}\mathrm{d}\Omega$$

3- Geiger-Marsden Experiment







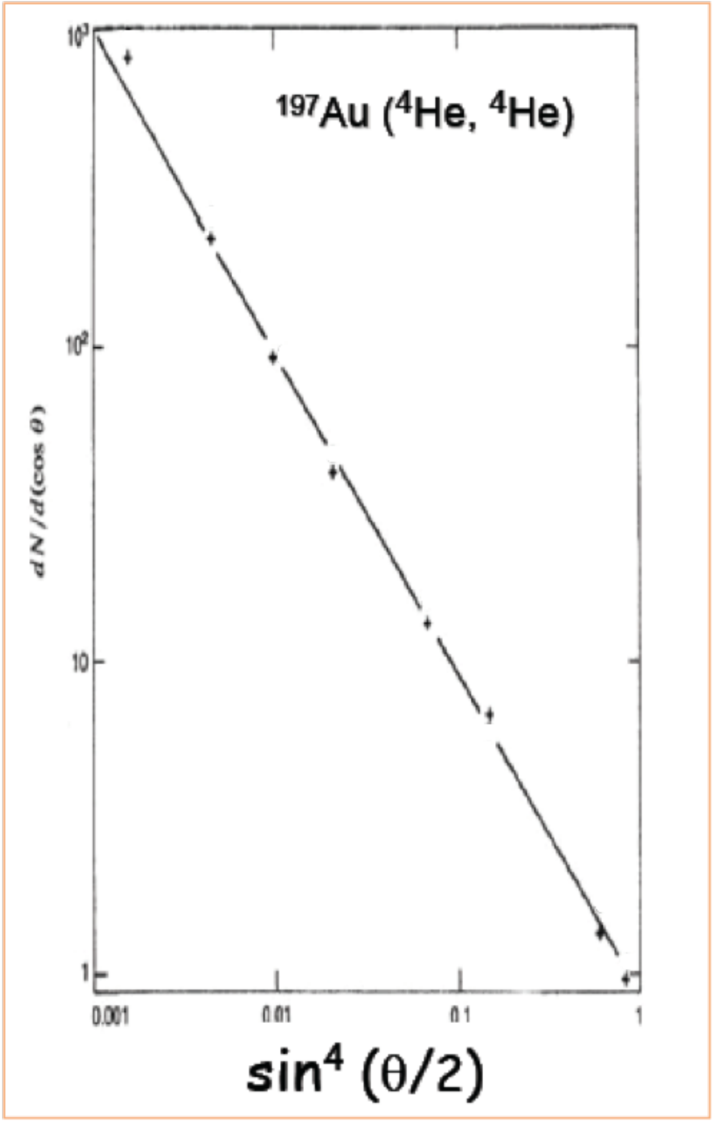
1909 : Elastic scattering of α-particles (z=2 and E_{kin}=5MeV) on Gold Nuclei (Z=79)

- Experience : H. Geiger and E. Marsden
- Interpretation : E. Rutherford

Classical mechanics : hyperbola trajectory, scattering angle depends on the impact parameter *b,* and the kinetic energy, E_{kin}, of the incident α-particle

$$\frac{\mathrm{d}\sigma}{\mathrm{d}\Omega}(\theta) = \frac{2\pi\, b\, \mathrm{d}b}{\mathrm{d}\Omega}$$

$$\boxed{\frac{\mathrm{d}\sigma}{\mathrm{d}\Omega}(\theta) = \left(\frac{zZ\alpha\hbar c}{4E_{kin}}\right)^2 \frac{1}{\sin^4\theta/2}}$$



Some experimental precautions :

- Beam:
  monoenergetic and isolated particles
- Target:
  thin in order to avoid multiple scattering
- Scattered particles:
  Statistical considerations

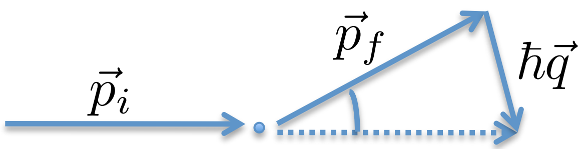
- Results compatible with the scattering by a point charge
- Atomic model with a central nucleus, radius $R < 4.5\ 10^{-14}$ m

4- Nuclear Size and density

- Focus on methods based on elastic diffusion of incident particles
- Electron scattering:
  - ☺: EM interaction
  - ☹: not sensible to neutron distribution
- Nuclear scattering:
  - ☺: sensible to neutron and proton distribution
  - ☹: sensible to strong interaction (and EM), not accurately described.

4.1- Scattering in QM

Consider a beam of particle scattering from a fixed potential *V(r).*



$$\vec{q} = \frac{\vec{p}_i - \vec{p}_f}{\hbar}$$

Starting from now, working in natural units : ħ=c=1

$$\vec{q} = \vec{p}_i - \vec{p}_f$$

Elastic scattering : a + b → a + b

only directions of momenta are changing : $\|\vec{p}_i\| = \|\vec{p}_f\| = p$

the target recoil is negligible.

$$\boxed{q = 2\,p\,\sin\frac{\theta}{2}}$$

Momentum transfer (units fm^{-1})

The scattering rate is characterized by the interaction cross-section

$$\sigma = \frac{\Gamma}{\Phi} = \frac{\text{Number of particles scattered per unit time}}{\text{Incident flux}}$$

Use Fermi's Golden Rule to calculate the transition rate : $\Gamma = 2\pi |M_{fi}|^2 \rho(E_f)$

Where M_{fi} is the matrix element and $\rho(E_F)$ is the density of final states.

We will use a first order perturbation theory (Born approximation), initial and final state are described by a plane wave (free particle).

The cross-section calculation requires :

1. Normalized wave functions
2. Matrix element in perturbation theory
3. Expression for incident flux
4. Expression for density of states

1. Normalized wave function:

Plane wave $\Longrightarrow$ solution of the form: $\psi = N e^{-i(Et - \vec{p}.\vec{r})}$

Normalize wave-functions to one particle in a box of side L:

$$|\psi|^2 = N^2 = 1/L^3$$

$$N = (1/L)^{3/2}$$

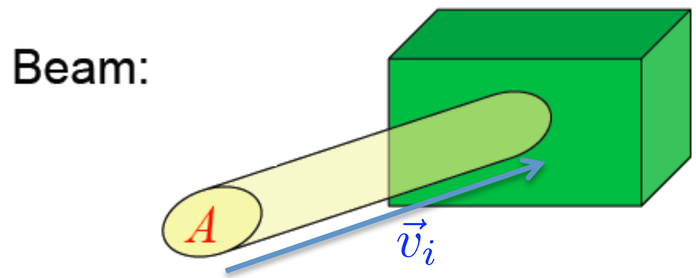
$$\psi = \frac{1}{L^{3/2}}\, e^{-i(Et - \vec{p}.\vec{r})}$$

2. Matrix Element: (contains the interesting physics of the interaction)

$$\begin{aligned} M_{fi} &= \langle \psi_f | \hat{H} | \psi_i \rangle = \int \psi_f^* \hat{H} \psi_i \, \mathrm{d}^3\vec{r} \\ &= \frac{1}{L^3} \int e^{-i\vec{p}_f \cdot \vec{r}} V(\vec{r}) e^{i\vec{p}_i \cdot \vec{r}} \mathrm{d}^3\vec{r} \\ &= \frac{1}{L^3} \int e^{i\vec{q}\cdot\vec{r}} V(\vec{r}) \mathrm{d}^3\vec{r} \end{aligned}$$

3. Incident flux:

Consider a "target" of area A and a beam of incident particles traveling at velocity v_i towards the target. Any incident particle within a volume $v_i A$ will cross the target area every second.



$$\Phi = v_i A \times n \times \frac{1}{A}$$

Where n is the number density of incident particles: $n = 1/L^3$

The flux is the number of incident particles crossing a unit area per unit time:

$$\Phi = \frac{v_i}{L^3}$$

4. Density of states:

for a box of side L, states are given by the periodic boundary conditions:

$$\vec{p} = (p_x, p_y, p_z) = \frac{2\pi}{L}(n_x, n_y, n_z)$$

Each state occupies a volume $(2\pi/L)^3$ in p space (neglecting spin).

Number of states between p and p+dp in solid angle dΩ

$$\mathrm{d}N = \left(\frac{L}{2\pi}\right)^3 \mathrm{d}^3p = \left(\frac{L}{2\pi}\right)^3 p^2 \mathrm{d}p\, \mathrm{d}\Omega$$

$$\rho(p) = \frac{\mathrm{d}N}{\mathrm{d}p} = \left(\frac{L}{2\pi}\right)^3 p^2\, \mathrm{d}\Omega$$

Density of states in energy :

$$E^2 = p^2 + m^2 \Rightarrow 2E\, \mathrm{d}E = 2p\, \mathrm{d}p \Rightarrow \frac{\mathrm{d}E}{\mathrm{d}p} = \frac{p}{E}$$

$$\rho(E) = \frac{\mathrm{d}N}{\mathrm{d}E} = \frac{\mathrm{d}N}{\mathrm{d}p}\frac{\mathrm{d}p}{\mathrm{d}E} = \left(\frac{L}{2\pi}\right)^3 p^2 \frac{E}{p} \mathrm{d}\Omega = \left(\frac{L}{2\pi}\right)^3 p\, E\, \mathrm{d}\Omega$$

Putting everything together:

$$d\sigma = \frac{1}{\Phi} 2\pi |M_{fi}|^2 \rho(E_f)$$

$$d\sigma = \frac{L^3}{v_i} 2\pi \left| \frac{1}{L^3} \int e^{i\vec{q}.\vec{r}} V(\vec{r})\, d^3\vec{r} \right|^2 \left(\frac{L}{2\pi}\right)^3 p_f\, E_f\, d\Omega$$

$$\boxed{\frac{d\sigma}{d\Omega} = \frac{1}{(2\pi)^2 v_i} \left| \int e^{i\vec{q}.\vec{r}} V(\vec{r})\, d^3\vec{r} \right|^2 p_f\, E_f}$$

- Non relativistic scattering:
  - Elastic diffusion : $p_i = p_f = m v_i$
  - $E_f \approx m$ ($p_f << m$)

$$\left.\frac{d\sigma}{d\Omega}\right|_{NR} = \frac{m^2}{(2\pi)^2} \left| \int e^{i\vec{q}.\vec{r}} V(\vec{r})\, d^3\vec{r} \right|^2$$

Scattering cross-section
in the Born approximation

- Relativistic scattering:
  - $v_i = c (=1)$
  - $p_f \approx E_f = E$

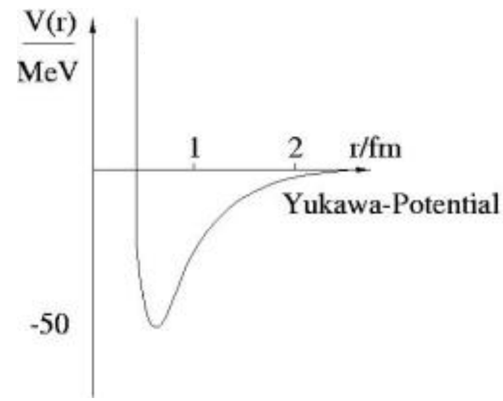
$$\left.\frac{d\sigma}{d\Omega}\right|_{R} = \frac{E^2}{(2\pi)^2} \left| \int e^{i\vec{q}.\vec{r}} V(\vec{r})\, d^3\vec{r} \right|^2$$

- Scattering form a Yukawa potential:
  (exchange of a massive scalar field, not seen in these lectures):

$$\text{Yukawa Potential}: V(\vec{r}) = g\frac{e^{-mr}}{r}$$

$$\int e^{i\vec{q}.\vec{r}} V(\vec{r})\, \mathrm{d}^3\vec{r} = \frac{4\pi g}{m^2+q^2}$$ (see demo 1)

$$\left.\frac{\mathrm{d}\sigma}{\mathrm{d}\Omega}\right|_R = \frac{4E^2g^2}{(m^2+q^2)^2}$$



- Scattering from a Coulomb potential:

$$\text{Coulomb Potential}: V(\vec{r}) = -\frac{e^2}{4\pi\epsilon_0 r} = -\frac{\alpha}{r}$$

i.e. a special case of Yukawa potential with g=-α and m=0

$$\int e^{i\vec{q}.\vec{r}} V(\vec{r})\, \mathrm{d}^3\vec{r} = -\frac{4\pi\alpha}{q^2}$$

- Non relativist coulomb scattering: Rutherford scattering

For Rutherford scattering, we are in the limit where the target recoil is neglected and the scattered particle is non-relativist.

Elastic scattering: $q = 2\,p\,\sin\dfrac{\theta}{2}$

$$\left.\frac{\mathrm{d}\sigma}{\mathrm{d}\Omega}\right|_{Rutherford} = \frac{m^2}{(2\pi)^2} \times \left(-\frac{4\pi\alpha}{q^2}\right)^2 = \frac{4m^2\alpha^2}{16p^4\sin^4\theta/2}$$

with $E_K = p^2/2m$

$$\left.\frac{\mathrm{d}\sigma}{\mathrm{d}\Omega}\right|_{Rutherford} = \frac{\alpha^2}{16E_K^2\sin^4\theta/2}$$

Reintroduction of ħ and c (via dimensional analysis):

$$\boxed{\left.\frac{\mathrm{d}\sigma}{\mathrm{d}\Omega}\right|_{Rutherford} = \left(\frac{\alpha\hbar c}{4E_K}\right)^2 \frac{1}{\sin^4\theta/2}}$$

- Relativist coulomb scattering: Mott scattering
  The limit where the target recoil is neglected and the scattered particle is relativist spin ½ is called Mott scattering.

  Elastic scattering: $q = 2\,p\,\sin\dfrac{\theta}{2} \simeq 2\,E\,\sin\dfrac{\theta}{2}$

$$\left.\frac{d\sigma}{d\Omega}\right|_R = \frac{E^2}{(2\pi)^2} \times \left(-\frac{4\pi\alpha}{q^2}\right)^2 = \frac{4E^2\alpha^2}{16E^4\sin^4\theta/2}$$

$$\left.\frac{d\sigma}{d\Omega}\right|_R = \frac{\alpha^2}{4E^2}\frac{1}{\sin^4\theta/2}$$

- For a relativistic spin ½ particle: use Dirac equation instead of Schrödinger.
  → there is an extra $\cos^2(\theta/2)$ factor due to the overlap between intial/final spin part of the wave-functions. Just Quantum Mechanics of spin ½.

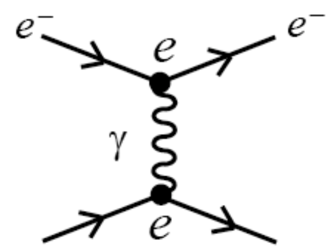
$$\left.\frac{d\sigma}{d\Omega}\right|_{\mathrm{Mott}} = \frac{\alpha^2}{4E^2}\frac{\cos^2\theta/2}{\sin^4\theta/2}$$

Non examinable

4.2- Electron scattering

Use electron as a probe to study deviations from a point-like nucleus:

- Electromagnetic interaction
- Coulomb potential: $V(\vec{r}) = -\frac{Z\alpha\hbar c}{r}$
- Mott scattering: $\frac{\mathrm{d}\sigma}{\mathrm{d}\Omega}\Big|_{\mathrm{Mott}} = \left(\frac{Z\alpha\hbar c}{2E}\right)^2 \frac{\cos^2\theta/2}{\sin^4\theta/2}$

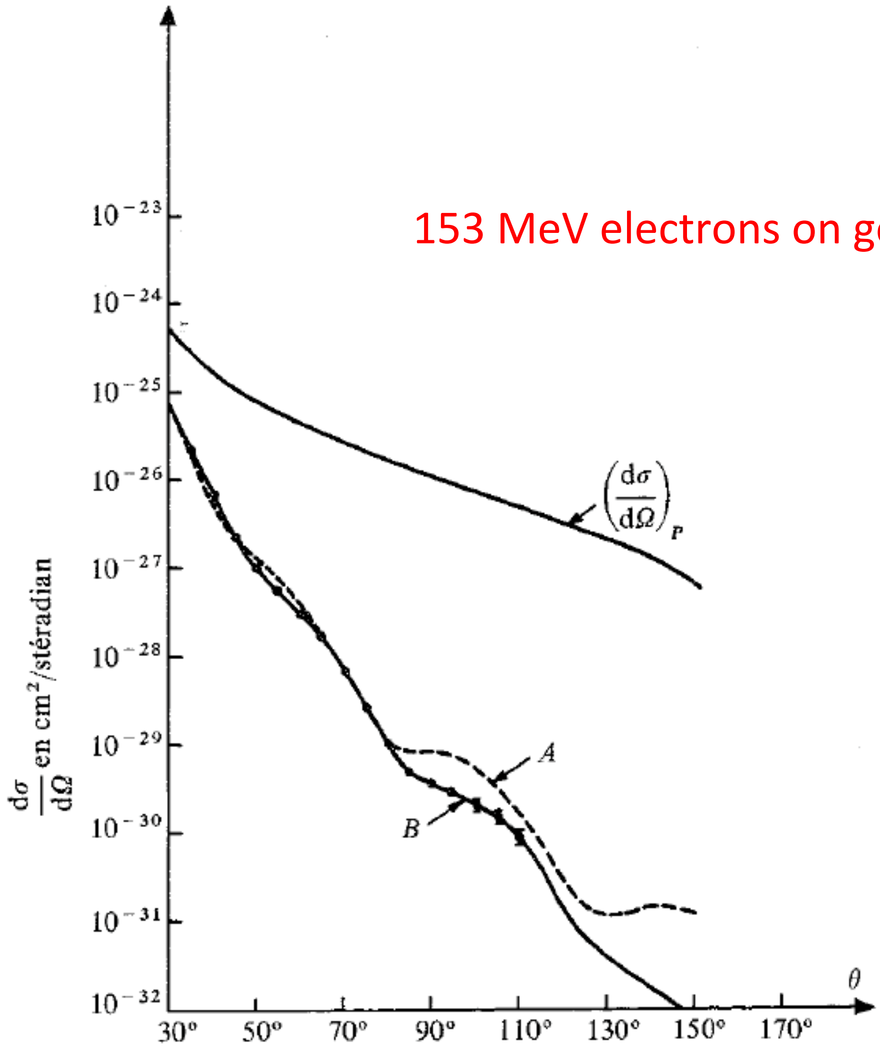


- To measure a distance of ≈1 fm, need energy

$$E = \frac{\hbar c}{\lambda} \simeq 200\ \mathrm{MeV}$$

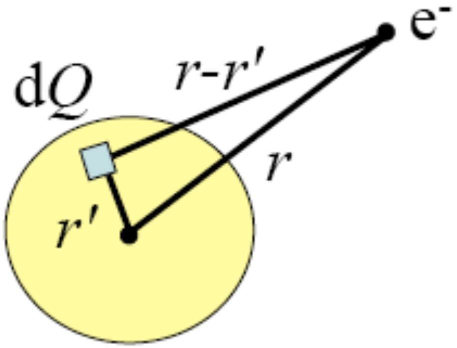
ultra-relativist domain

- Theory does not match experiment !



4.3- Scattering with en extended nucleus

$V(\vec{r})$ depends on the distribution of charge in nucleus



Potential energy of electron due to charge d*Q:*

$$\mathrm{d}V = -\frac{e\,\mathrm{d}Q}{4\pi\epsilon_0|\vec{r}-\vec{r'}|}$$

where $\mathrm{d}Q = Ze\rho(\vec{r'})\mathrm{d}^3\vec{r'}$

Charge distribution $\rho(\vec{r})$ normalized to unity

So $$V(\vec{r}) = -\int \frac{e^2\,Z\rho(\vec{r'})}{4\pi\epsilon_0|\vec{r}-\vec{r'}|}\mathrm{d}^3\vec{r'} = -Z\alpha\int\frac{\rho(\vec{r'})}{|\vec{r}-\vec{r'}|}\mathrm{d}^3\vec{r'}$$

This formula is the convolution of the Coulomb potential Zα/r with the charge distribution ρ(r)

In order to evaluate the matrix element M_{fi}:

$$M_{fi} = \int e^{i\vec{q}.\vec{r}}\, V(\vec{r})\, \mathrm{d}^3\vec{r}$$

We can use the convolution theorem : FT(f*g)=FT(f)×FT(g),

in order to evaluate this matrix element

$$M_{fi} = -Z\alpha \int \frac{e^{i\vec{q}.\vec{r}}}{r}\, \mathrm{d}^3\vec{r} \int \rho(\vec{r}) e^{i\vec{q}.\vec{r}}\, \mathrm{d}^3\vec{r}$$

$$\frac{\mathrm{d}\sigma}{\mathrm{d}\Omega} = \left(\frac{\mathrm{d}\sigma}{\mathrm{d}\Omega}\right)_{\mathrm{Rutherford}} |F(q)|^2$$

where $\boxed{F(q) = \int \rho(\vec{r}) e^{i\vec{q}.\vec{r}}\, \mathrm{d}^3\vec{r}}$ is called the FORM FACTOR and is the Fourier transform of the charge distribution

This can be generalized by: $\frac{\mathrm{d}\sigma}{\mathrm{d}\Omega} = \left(\frac{\mathrm{d}\sigma}{\mathrm{d}\Omega}\right)_{\mathrm{Mott}} |F(q)|^2$

In the case of a spherical symmetry, ρ=ρ(r), a simple calculation shows that

$$F(q) = \int_0^\infty \rho(r) \frac{\sin qr}{qr} 4\pi r^2 \mathrm{d}r$$

(see demo 2)

and

$$\rho(r) = \frac{1}{2\pi^2} \int_0^\infty F(q) \frac{\sin qr}{qr}\, q^2 \mathrm{d}q$$

So the experimental method is:

1. measure the cross-section.
2. infer F(q).
3. Fourier transform to get the charge distribution.

Experimentally, it is really difficult to measure the cross-section for the high values of q (very small cross sections), so the last steps are often:

3. Assume a parameterization for the nuclear density →Fourier transform to get a "theoretical" F (q)
4. Compare "theoretical" and "experimental" F(q) to get the parameters of the charge distribution.

- So far, we have not considered the spin of the target.
- A more refined description, takes into account a magnetic form factor due the magnetic moments distribution in the nucleus

$$\frac{\mathrm{d}\sigma}{\mathrm{d}\Omega} = \left(\frac{Z\alpha\hbar c}{2E}\right)^2 \frac{1}{\sin^4\theta/2}\left(|F_e(q)|^2\cos^2\theta/2 - \frac{q^2}{2m^2}|F_m(q)|^2\sin^2\theta/2\right)$$

$$\frac{\mathrm{d}\sigma}{\mathrm{d}\Omega} = \left(\frac{\mathrm{d}\sigma}{\mathrm{d}\Omega}\right)_{\mathrm{Mott}}\left[|F_e(q)|^2 - \frac{q^2}{2m^2}\tan^2\theta/2|F_m(q)|^2\right]$$

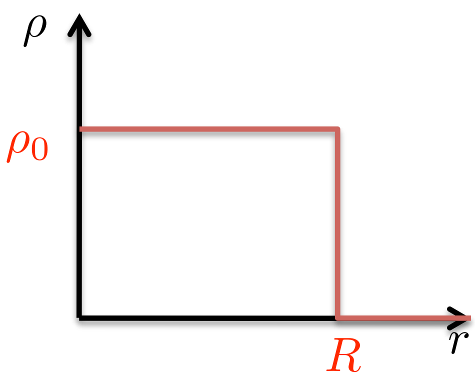
- Other improvement : Nuclear recoil

$$\frac{\mathrm{d}\sigma}{\mathrm{d}\Omega} = \left(\frac{\mathrm{d}\sigma}{\mathrm{d}\Omega}\right)_{\mathrm{Mott}}\frac{E_f}{E_i}\left[|F_e(q)|^2 - \frac{q^2}{2m^2}\tan^2\theta/2|F_m(q)|^2\right]$$

Non examinable

4.4 Charge density

- The most simple way to describe the charge density is a sharp edge distribution (hard sphere):



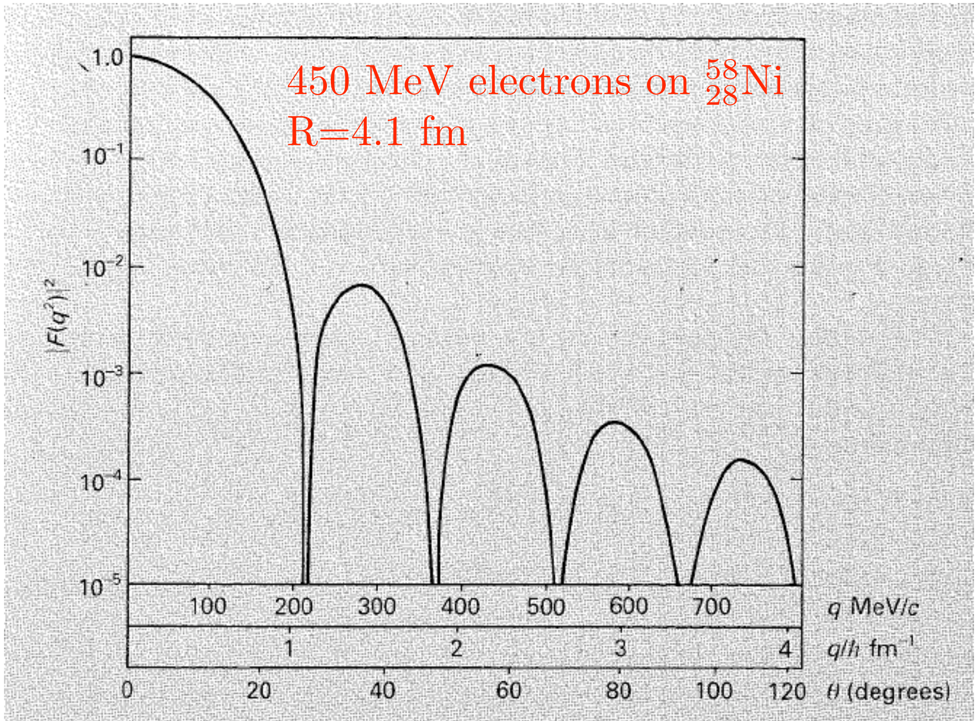


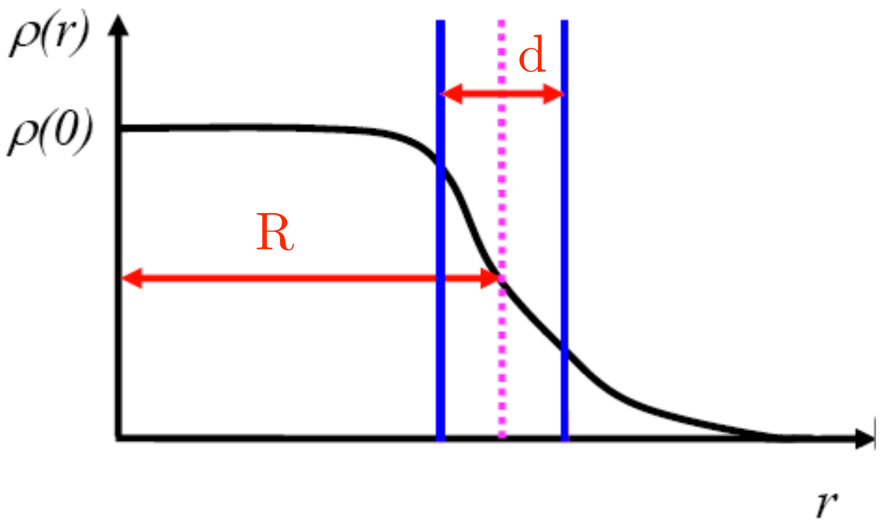
Fig. 3.3 The square of the form factor $|F(q^2)|^2$ as a function of q for a model I nucleus having $a = 4.1$ fm. The abscissa is also marked in inverse fermis ($q/\hbar$) and in degrees for an angle of scatter at a fixed nucleus for incident electrons of 450 MeV. Note that the ordinate is logarithmic.

- In this case, the form factor is:

$$F(q) = \frac{3(\sin x - x\ \cos x)}{x^3}$$

with $x=qR$
and has the following shape:
It is in fact the spherical Bessel function $j_1(x)$.
First zeros: $x=1.43\times\pi$, $x=2.46\times\pi$

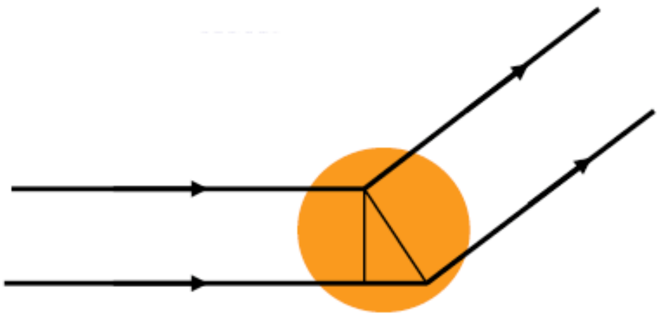
- A more realistic parameterization of the nuclear charge density is a Fermi-like parameterization:



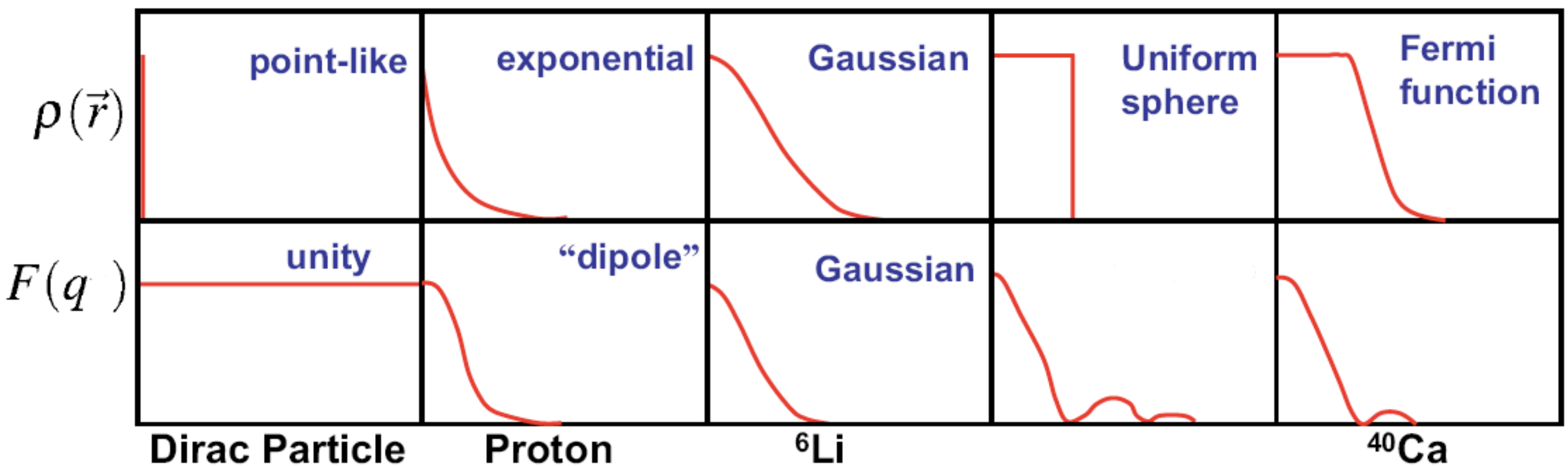
$$\rho(r) = \frac{\rho(0)}{1 + e^{\left(\frac{r-R}{d}\right)}}$$

- *R* is the "radius" of the nucleus. $\rho(R)=\rho(0)/2$
- *d* governs the drop of the density at the surface.
  $\rho(R-d)=0.62\times\rho(0)$ and $\rho(R+d)=0.38\times\rho(0)$
  The 90% to 10% thickness is $4.39\times d$

There is nothing mysterious about form factors – similar to diffraction of plane waves in optics.

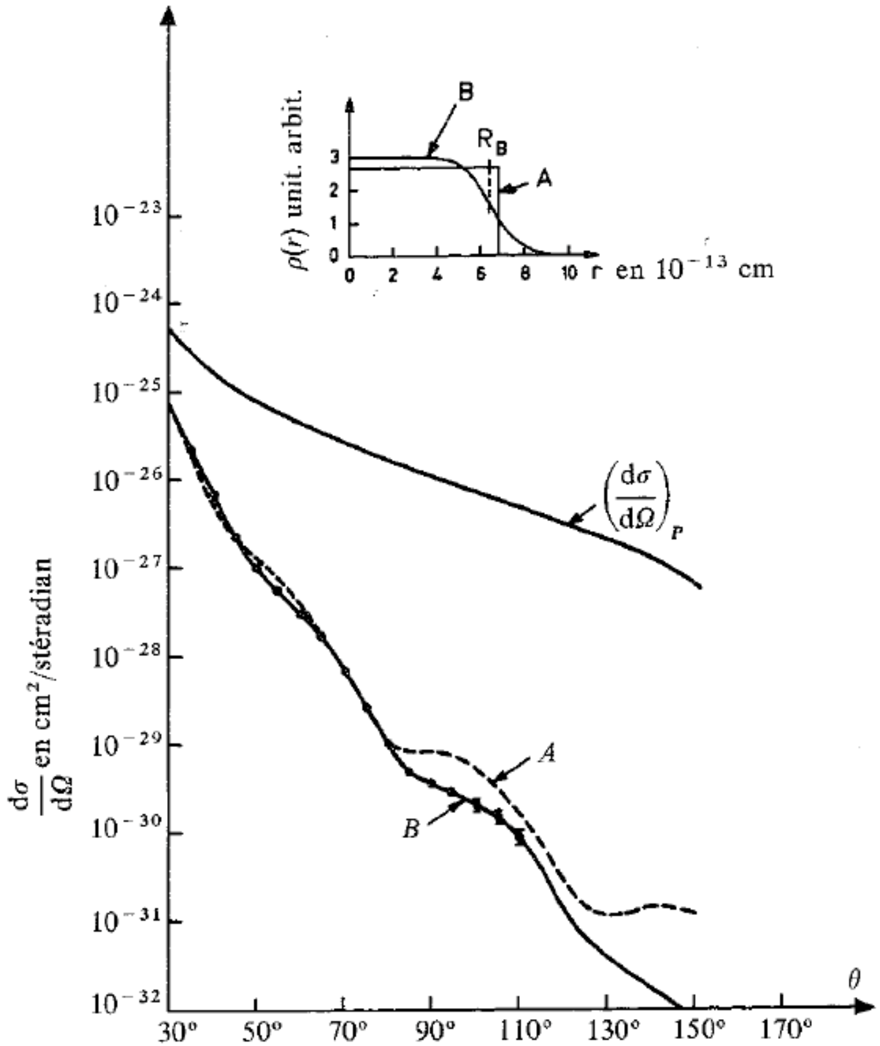


The finite size of the scattering centre introduces a phase difference between plane waves "scattered from different points in space".



•**NOTE that for a point charge the form factor is unity.**

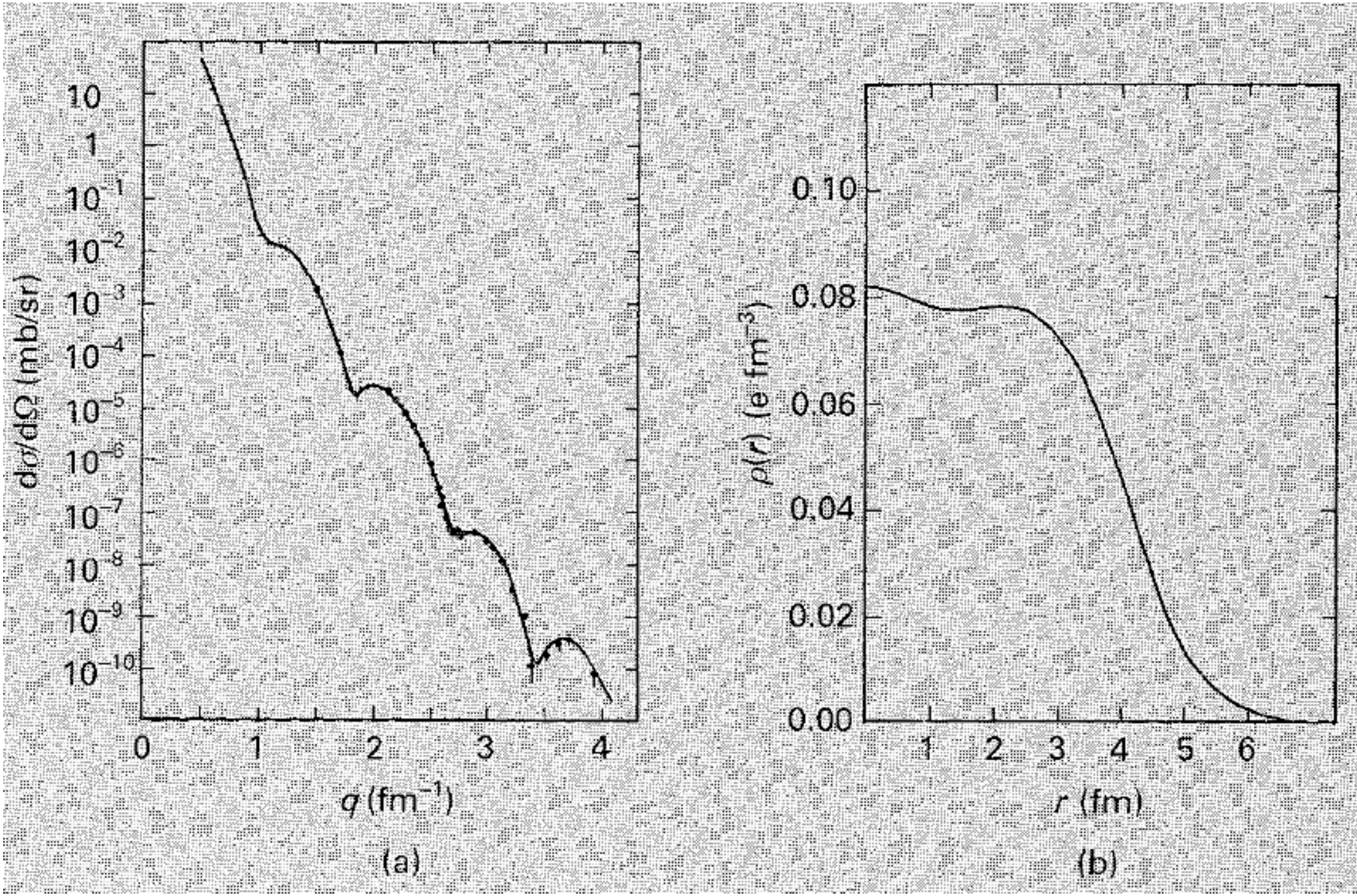
4.5 Experimental results



153 MeV electrons on gold (Z=79)

FIGURE I.8

Distribution angulaire de la diffusion élastique d'électrons de 153 MeV sur l'or. On constatera que l'approximation, $(d\sigma/d\Omega)_p$, de charge ponctuelle n'est pas réaliste. Le meilleur accord avec l'expérience est obtenu dans l'hypothèse *B* d'une distribution de charge $\rho(r)$ à bord diffus (courbe théorique en trait plein passant au mieux par les points expérimentaux).



$450\ \mathrm{MeV\ electrons\ on}\ {}^{58}_{28}\mathrm{Ni}$

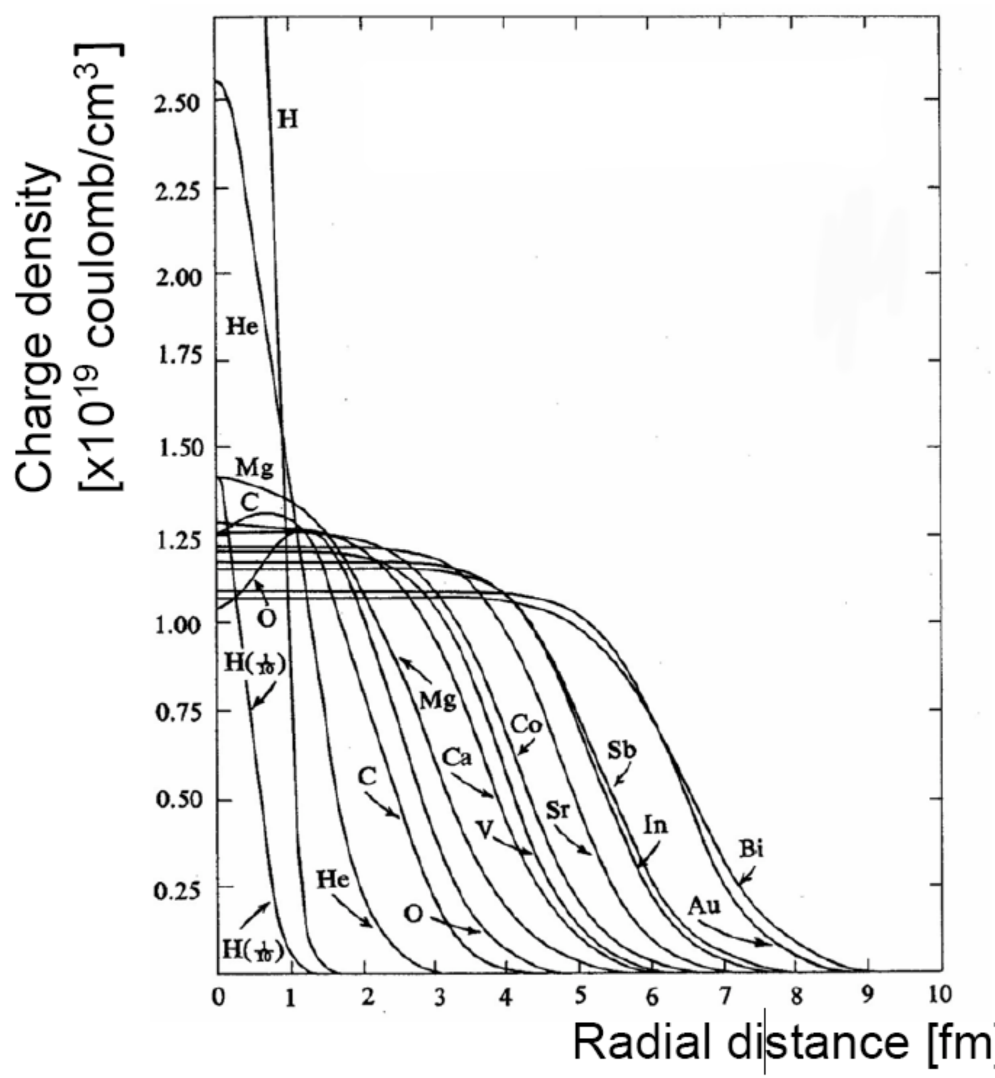
Using the Fermi parameterization:

$$\rho(r) = \frac{\rho(0)}{1 + e^{\left(\frac{r-R}{d}\right)}}$$

For all nuclei

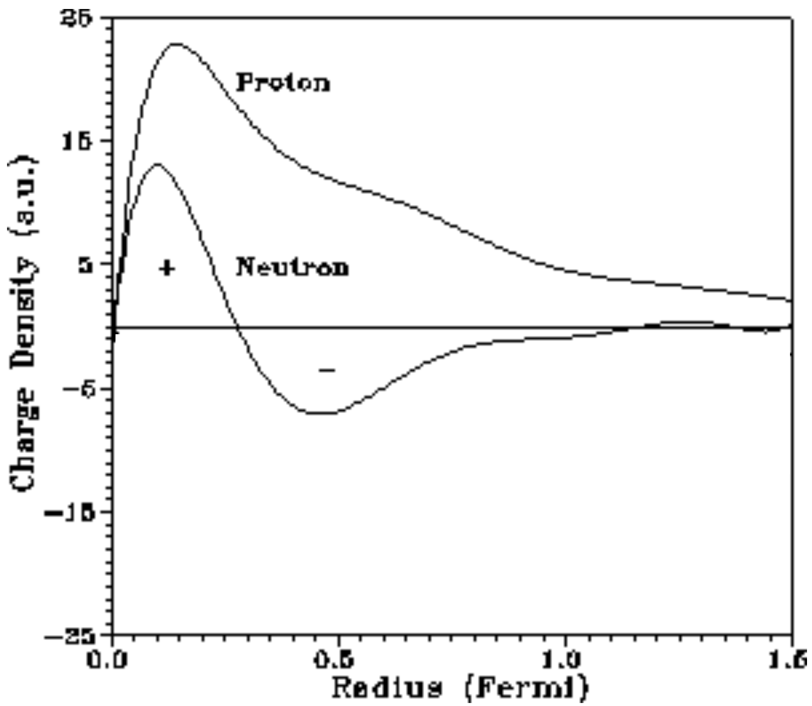
- $R \approx r_0 A^{1/3}$ with $r_0 = 1.2$ fm (central density almost identical for all nuclei)
- Nuclear density: constant for all nuclei $\rho_0 \approx 0.08$ nucleons/fm^3 saturation of nuclear forces → short distance interaction
- $d \approx 2.5$ fm (governed by the range of the nuclear interaction)

e$^-$ scattering data



Aside : Nucleon charge density

- *Scattering experiments of high energy electrons*



Non examinable

4.6- Nuclear density

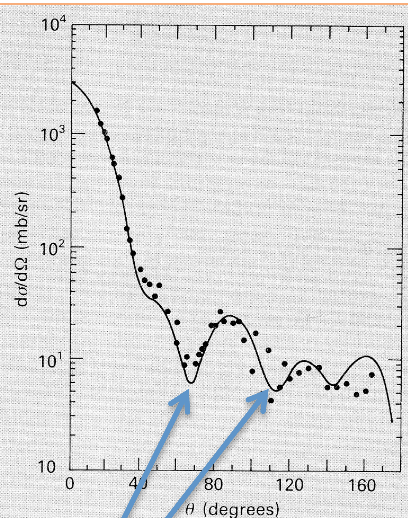
- Up to now, we have used electromagnetic probes
  → charge density → charge radius
- Use a nuclear projectile to probe the nuclear density:
  n, p, π (The charge distribution in neutron and proton is not homogenous)
  more complicated due to the unknown expression of the nuclear force → optical model of the nucleus

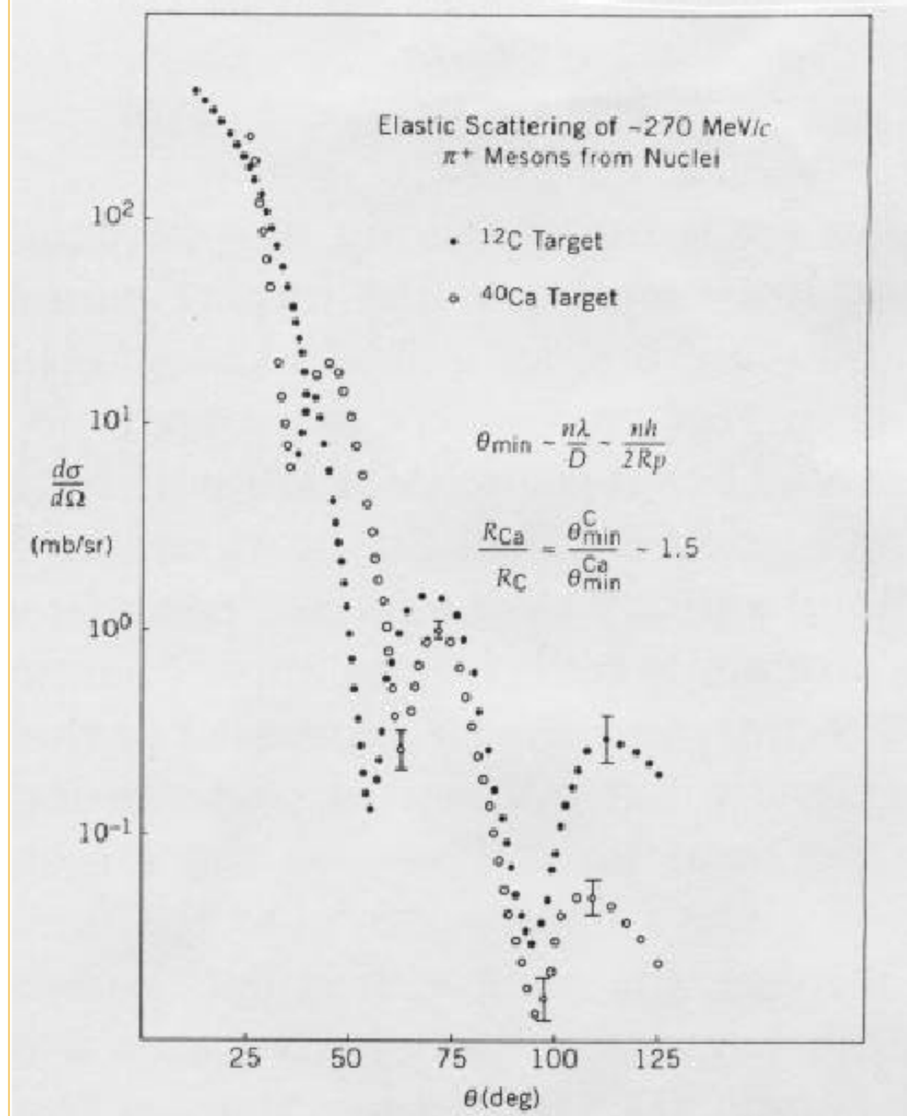
$V(r) = \quad V_c(r)$	Coulomb potential for charged particles only
$-V\, f_1(r)$	The nuclear potential well
$-i\, W\, f_2(r)$	The imaginary potential representing absorption of the incident nucleon
$+V_{LS}$	The spin-orbit interaction

$f_i(r)$ are often chosen as fermi-like distribution

Non examinable

Fig. 3.6 The measured differential cross-section (solid points) against centre-of-mass angle for the elastic scattering of 14 MeV neutrons by nickel. The solid line is a prediction from a set of optical model parameters. The momentum transfer at the diffraction minima at 68° and 110° are about 179 MeV/c and 262 MeV/c ($q/\hbar = 0.91$ and $1.33\ \mathrm{fm}^{-1}$) respectively. They do not coincide with the minima in the results for electron scattering (Fig. 3.4) because the absorption of the neutron wave by nuclear matter alters the effective diffracting aperture from that in electron scattering. (Figure adapted from Bauer *et al.*, 1963. The errors on the measured points have not been included. The relative errors vary from 5% at small angles to 15% at large angles. There is an overall uncertainty of 10% in the absolute cross-sections.)





"Holes" arising from the scattering from an absorbing object.

Non examinable

Summary

- The total density of the nucleus is the sum of the neutron and the proton density.
- Experimental results suggest, the same shape for neutron and proton density.

$$\frac{\rho_Z}{\rho_N} = \frac{Z}{N}$$

$$\rho_{\mathrm{tot}} = \rho_Z\left(1 + \frac{N}{Z}\right)$$

- If N=Z (symmetric nuclear matter),

$$\rho_{\mathrm{tot}} = 2\rho_0 = 0.17\ \mathrm{nucleons/fm}^3$$

- The radius is : $R \approx r_0 A^{1/3}$ with $r_0 \approx 1.2$ fm
  Hard sphere approx: $r_0 = (3/4\pi\rho_{tot})^{1/3} \approx 1.1$ fm

- In a neutron star: $\rho \approx 10^{38}$ neutrons/cm^3 ≈ 2 10^{11} kg/cm^3

4.7- Nuclear radii from Muonic atoms

Muons brought to rest in matter, get trapped in atomic orbit and have a higher probability than electrons of spending time inside the nucleus

Bohr radius	Energy
~1/Zm	~Z^2m

μ mass	μ lifetime
~207 m_e	2 μs

The muons make transitions to low energy levels, emitting X-rays, before decaying

- For hydrogen with electrons: $r=a_0 = 5\times10^4$ fm (Bohr radius)
- For lead with muons : $r = a_0/(Zm)=(5\times10^4)/(82\times207)=3$ fm
  to be compared to the radius of the lead : $R = r_0\, A^{1/3} = 1.2\times208^{1/3} \sim 7$ fm
  → the muons spend a large fraction of their time inside the nucleus.

Transition energy (2p→1s) : 16.41 MeV (Bohr theory) vs 6.02 (measured)

⇒$Z_{effective}$ and E are changed relative to electrons ⇒ Radius measurement

4.8- Nuclear Radii from Mirror nuclei

- Mirror nuclei : Mirror nuclei are a pair of nuclei where the number of protons and neutrons are interchanged
(e.g. ${}^{11}_{6}\mathrm{C}$ and ${}^{11}_{5}\mathrm{B}$)
- Mirror nuclei have different masses due to the p-n difference and the different coulomb terms in the binding energy:

$$M(A,Z+1)-M(A,Z)=\Delta E_c+m_p+m_e-m_n$$

where ΔE_c is the coulomb energy difference between the two nuclei.
A classical calculation, where the nucleus is considered to be analogous to a uniformly charged sphere of radius R, shows that

$$\Delta E_c=\frac{3\alpha}{5}\frac{(2Z+1)}{R}$$

So the radius can be determined from the mass difference between the two atoms. This mass difference can be measured from the β^+ decay spectrum of the (A,Z+1) member of the pair:

$$ {}^{11}_{6}\mathrm{C} \longrightarrow {}^{11}_{5}\mathrm{B} + e^+ \; + \; \nu_e $$

as we have seen : $M(^{11}\mathrm{C}) - M(^{11}\mathrm{B}) = 2m_e + K_\beta^{\mathrm{max}}$

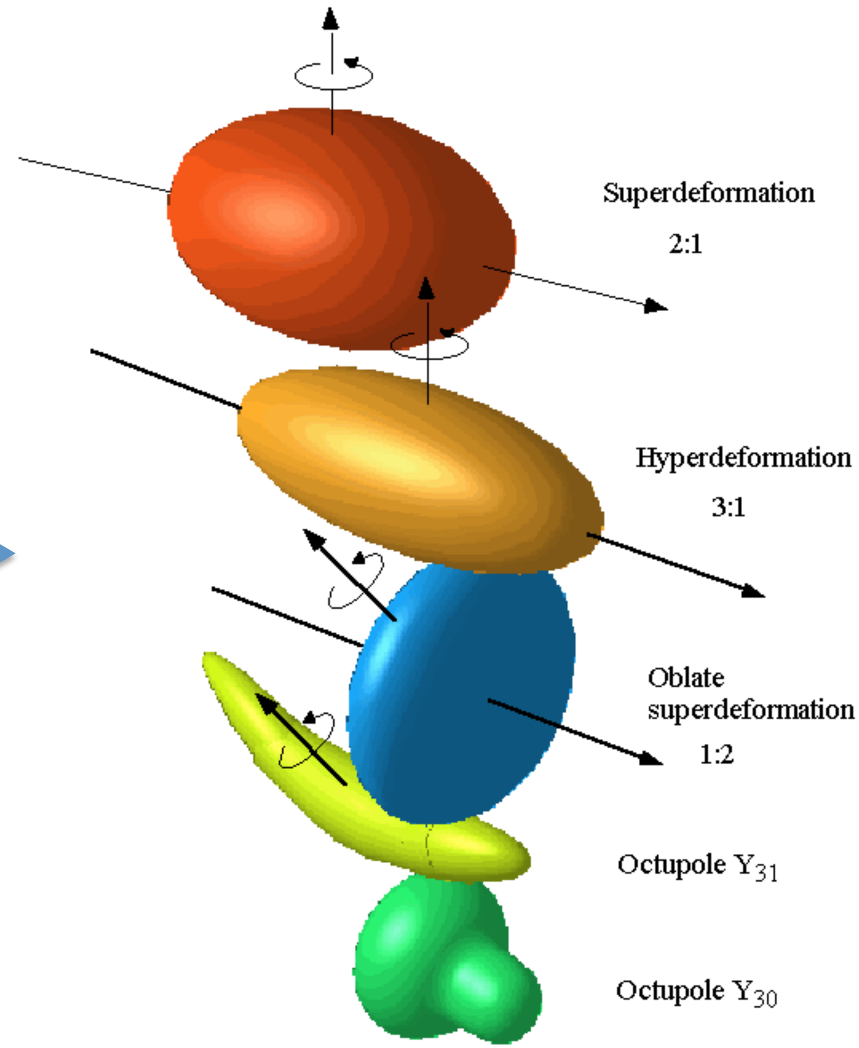
The radius is then:

$$ R = \frac{3\alpha(2Z+1)}{5}\left[\frac{1}{K_\beta^{\mathrm{max}} - m_p + m_e + m_n}\right] $$

5- Nuclear shape

The nucleus are not necessarily spherical

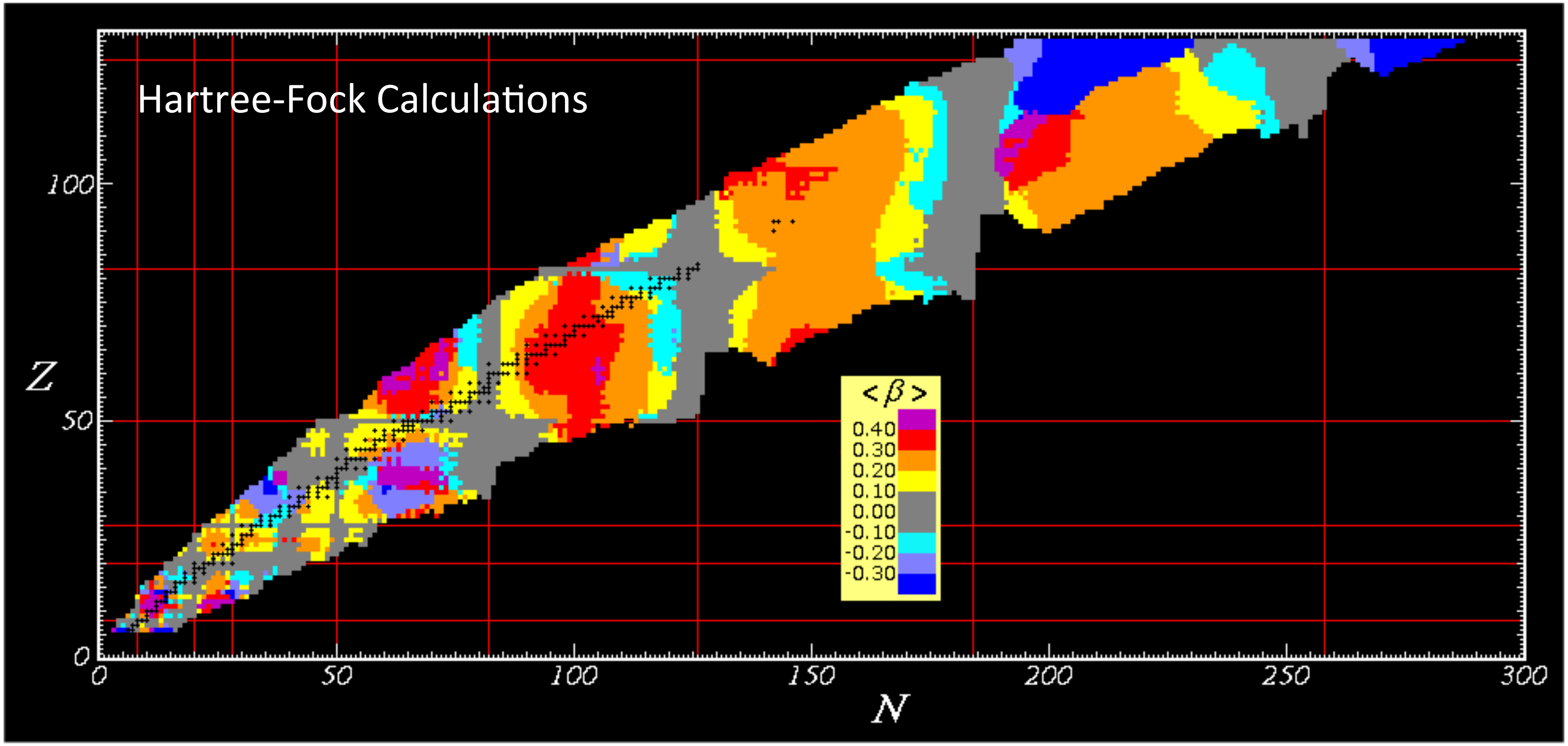
Extreme deformations



Non examinable

Deformation of nuclei

(β is related to Q)



5.1- Radius mean square

- For a nucleus with a hard sphere density:

$$< r^2 >= \frac{\int_0^\infty \rho\, r^2 4\pi r^2\, \mathrm{d}r}{\int_0^\infty \rho\, 4\pi r^2\, \mathrm{d}r} = \frac{\int_0^R r^4\, \mathrm{d}r}{\int_0^R r^2\, \mathrm{d}r} = \frac{3}{5} R^2$$

  For example, the analysis of hyperfine transitions show an identical $<r^2>$ value for ^{180}Hg and ^{205}Hg, although the different nucleon number should give different values. These values can be explained by a different deformation.

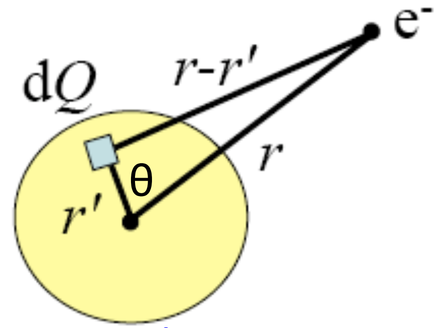
- In general, light nuclei are spherical, heavy nuclei are deformed (magic numbers excepted)

5.2- Nuclear moments

- Static electromagnetic properties of nuclei are specified in terms of electromagnetic moments, which give information about the way magnetism and charge is distributed throughout the nucleus.
- The two most important moments are:
  - Electric Quadrupole Moment: Q
  - Magnetic Dipole Moment: μ
- Electric moments:
  Depends on the charge distribution inside the nucleus and are a measure of Nuclear shape (contours of constant charge density)
  - Q=0 : spherical nucleus
  - Large Q : Highly deformed nucleus

The nuclear shape is parameterized by a multipole expansion of the external field or potential.

$$V(\vec{r}) = -\frac{1}{4\pi}\int \frac{\rho(\vec{r'})}{|\vec{r}-\vec{r'}|}\mathrm{d}^3\vec{r'} \qquad \int \rho(\vec{r'})\mathrm{d}^3\vec{r'} = Ze$$



r=distance to observer
r′=distance to charge element

$$|\vec{r}-\vec{r'}| = [r^2 + r'^2 - 2rr'\cos\theta]^{1/2}$$

$$\Rightarrow |\vec{r}-\vec{r'}|^{-1} = r^{-1}[1 + r'^2/r^2 - 2\tfrac{r'}{r}\cos\theta]^{-1/2}$$

$$\Rightarrow |\vec{r}-\vec{r'}|^{-1} = r^{-1}\left[1 - \tfrac{1}{2}\left(\tfrac{r'^2}{r^2} - 2\tfrac{r'}{r}\cos\theta\right) + \tfrac{3}{8}\left(\tfrac{r'^2}{r^2} - 2\tfrac{r'}{r}\cos\theta\right)^2 + ...\right]$$

$$\Rightarrow |\vec{r}-\vec{r'}|^{-1} \simeq r^{-1}\left[1 + \tfrac{r'}{r}\cos\theta + \tfrac{1}{2}\tfrac{r'^2}{r^2}(3\cos^3\theta - 1) + ...\right]$$

Expansion in powers of *r′/r* (Legendre polynomials).

$$V(r) = \frac{1}{4\pi r}\left[Ze + \frac{1}{r}\int r'\cos\theta\,\rho(r')\mathrm{d}^3\vec{r'} + \frac{1}{2r^2}\int r'^2(3\cos^2\theta - 1)\,\rho(r')\mathrm{d}^3\vec{r'} + ...\right]$$

Let *r* define *z*-axis: $z = r' \cos\theta$

$$V(r) = \frac{1}{4\pi r}\left[Ze + \frac{1}{r}\int z\,\rho(r')\mathrm{d}^3\vec{r'} + \frac{1}{2r^2}\int(3z^2 - r'^2)\,\rho(r')\mathrm{d}^3\vec{r'} + \ldots\right]$$

Quantum limit: $\rho(\vec{r'}) = |\psi(\vec{r'})|^2$

- E0 Moment: $\int \psi^*\psi\ \mathrm{d}^3r'$ charge
- E1 Moment: $\int \psi^* z\psi\ \mathrm{d}^3r'$ electric dipole
- E2 Moment: $\int \psi^*(3z^2 - r'^2)\psi\ \mathrm{d}^3r'$ electric quadrupole

Nuclear wavefunctions have definite parity: $|\psi(\vec{r})|^2 = |\psi(-\vec{r})|^2$

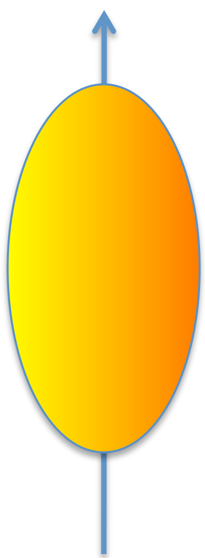
⇒ Electric dipole is always zero.

Electric Quadrupole Moment:

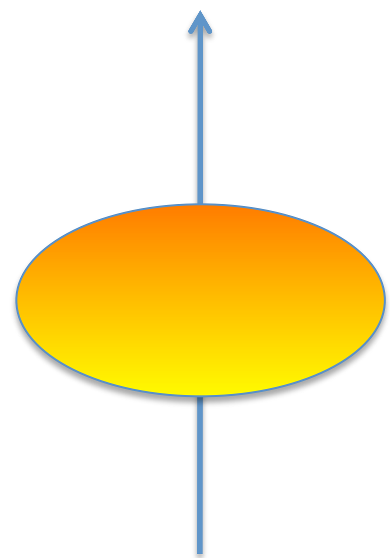
$$Q = \frac{1}{e} \int \psi^* (3z^2 - r^2) \psi \, \mathrm{d}^3 r$$

Units: m^2 or barns

If spherical symmetry, $\langle z^2 \rangle = 1/3 \langle r^2 \rangle \rightarrow Q=0$



$Q>0 \rightarrow$ Prolate:
"Rugby balloon" like



$Q<0 \rightarrow$ Oblate:
"Flying saucer" like

- A non-zero quadrupole moment Q indicates that the charge distribution is not spherically symmetric.
- A non spherical nucleus will have rotational states of motion which by their regularity, are identifiable in the spectrum of the excited states.
- Electric quadrupole moments of nuclei can be measured from hyperfine splitting of atomic spectral lines, from the rotational spectra, and other spectroscopic techniques.