Chapter 6

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Electromagnetic interactions

Outline/Plan

1. Cross-section

2. Matrix element

3. Quantum ElectroDynamics (QED) Feynman rules

1. Section efficace

2. Elément de matrice

3. Rêgles de Feynman pour l'électro-dynamique quantique

1- Cross-section

Consider the general process for two-particle scattering :

$$a+b \rightarrow 1+2+...+n$$

- States normalization : in NRQM usually one takes 1 particle per unit volume. For KG solutions :

$$\phi_a = N_a e^{-ip_a x} \Rightarrow \rho_a = i\left[\phi_a^*(\partial_t \phi_a) - (\partial_t \phi_a^*)\phi_a\right] = 2N_a^2 E_a$$

- Usual prescription (covariant normalization) :

2E particles per unit volume

$$\int_V d^3x \rho_a = 2E_a \Rightarrow N_a = V^{-1/2}$$

1- Cross-section

The general expression of the cross-section reads :

$$d\sigma = \frac{1}{2E_a 2E_b |\vec{v}_a - \vec{v}_b|} \int (2\pi)^4 \delta^4(p_f - p_i) \overline{|T_{fi}|^2} \prod_{k=1}^{k=n} \frac{d^3 p_k}{(2\pi)^3 2E_k}$$

- $2E_a 2E_b |\vec{v}_a - \vec{v}_b| / V^2$ is the flux factor (number of incident particles per unit area reaching the target per unit time $\times$ number of target particles) with normalization at *2E* particles per unit volume.

- $\prod_{k=1}^{k=n} \frac{V d^3 p_k}{(2\pi)^3 2E_k}$ is the final state available phase space (for one particle in a volume *V* with a momentum in *d³p* one has the usual expression : $V d^3 p / (2\pi)^3$

1- Cross-section

- $(2\pi)^4 \delta^4(p_f - p_i)\overline{|T_{fi}|^2}$ is the transition rate per volume unit. It is the part containing the physics of the interaction processes. Each diagram's line is associated with a normalization factor N_i (i=a,b,1,…,n) which induces an overall factor :

$$\left[\underbrace{(V^{-1/2})^2}_{a,b} \times \underbrace{(V^{-1/2})^n}_{1...n} \right]^2 = V^{-(n+2)}$$

balancing the contributions of the flux and the phase space => the cross-section is independent of the normalization volume!

1- Cross-section

Application : Rutherford scattering formula (2+2 process => n=2).

- Flux factor : $2E_a 2E_b \left| \vec{v}_a - \vec{v}_b \right| = 4\left[(p_a.p_b)^2 - m_a^2 m_b^2 \right] = 4\left| \vec{p}_a^* \right| \sqrt{s}$

  where the Mandelstam variable s has been used $s = \left(p_a + p_b \right)^2$

- 2-bodies phase space :
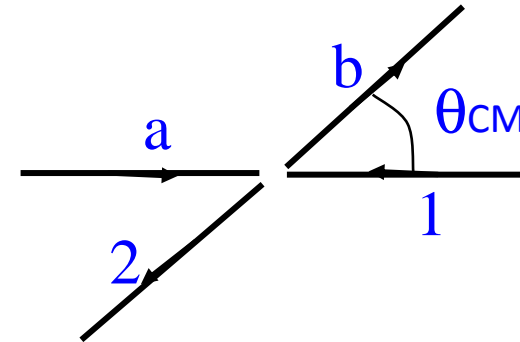
$$dQ_2 = \int_4 (2\pi)^4 \delta^4 (p_f - p_i) \frac{d^3 p_1}{(2\pi)^3 2E_1} \frac{d^3 p_2}{(2\pi)^3 2E_2}$$

$$= \frac{1}{16\pi^2} \frac{\left| \vec{p}_1 \right|^3 d\Omega_1}{\left| \vec{p}_1 \right|^2 E_T - \vec{p}_1.\vec{p}_T E_1} \quad \text{with } (p_T) = (p_a + p_b)$$

$$= \frac{1}{16\pi^2} \frac{\left| \vec{p}_1^* \right| d\Omega_1}{\sqrt{s}}$$

1- Cross-section

Application : Rutherford scattering formula (2+2 process => n=2).

- Transition rate :

$$|T_{fi}|^2 = \left(\frac{e^2}{q^2}\right)^2 (p_a + p_b).(p_1 + p_2)$$

- Mandelstam variables :

$$s = (p_a + p_1)^2 \quad t = (p_a - p_b)^2 \quad u = (p_a - p_2)^2$$

$$|T_{fi}|^2 = \left(\frac{4\pi\alpha}{t}\right)^2 (s - u)^2 \quad \text{and} \quad \frac{d\sigma}{d\Omega} = \frac{1}{64\pi^2 s}\frac{|\vec{p}_1^{\,*}|}{|\vec{p}_a^{\,*}|}|T_{fi}|^2$$

- In the LAB frame (target at rest)

1- Cross-section

Application : Rutherford scattering formula (2+2 process => n=2).

- "Classical" case : a is a light particle (negligible m), 1 at rest initially => use of the LAB frame

$$p_a \simeq \left(\left| \vec{k} \right|, \vec{k} \right) \quad p_1 \simeq \left(M, \vec{0} \right)$$

$$t \simeq -4 \left| \vec{k} \right|^2 \sin^2(\theta/2) \quad (s-u)^2 = 16 \left| \vec{k} \right|^2 M^2$$

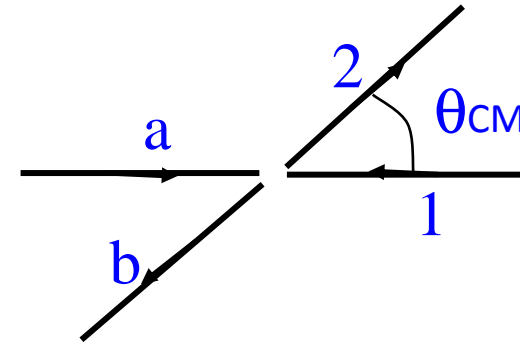$$4\left[(p_a \cdot p_1)^2 - m^2 M^2 \right] \simeq 4 \left| \vec{k} \right|^2 M^2$$

- With the fine structure constant $\alpha = e^2 / 4\pi$

  one gets the standard Rutherford formula :

$$\frac{d\sigma}{d\Omega} = \frac{\alpha^2}{4 \left| \vec{k} \right|^2 \sin^4(\theta/2)}$$

1- Cross-section

Question : how one computes the transition rate

$$\left| T_{fi} \right|^2 = \left(\frac{e^2}{q^2} \right)^2 (p_a + p_1) \cdot (p_b + p_2)$$
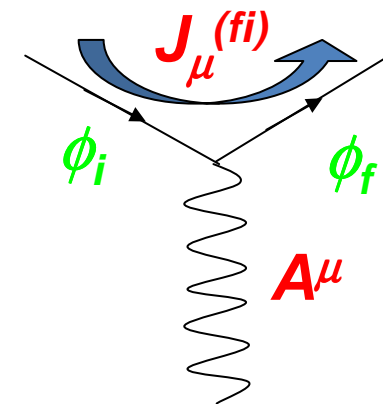

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2- Matrix element

- Classical theory : transition amplitude (covariant expression) from initial *(i)* to final *(f)* state under the action of a perturbation potential *V*.

$$T_{fi} = -i \int d^4x \, \phi_f^*(x) V(x) \phi_i(x)$$

- 1st order K.G. potential : $V = -ie\left(\partial_\mu A^\mu + A^\mu \partial_\mu\right)$
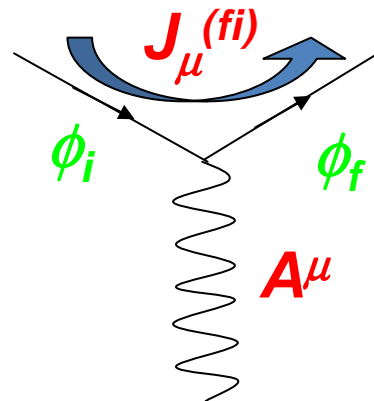
- Integrating by parts introduces the transition current :

$$T_{fi} = -i \int d^4x \, A^\mu \underbrace{(-ie)\left[\phi_f^* \partial_\mu \phi_i - \partial_\mu \phi_f^* \phi_i\right]}_{J_\mu^{(fi)}}$$

$J_\mu^{(fi)}$

ϕ_i ϕ_f

A^μ

2- Matrix element

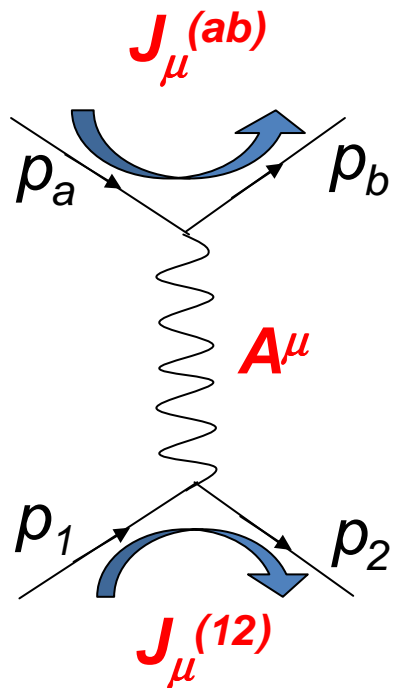- *(i)* and *(f)* are taken as free particle solutions : $\phi_{i,f}(x) = N_{i,f} e^{-ip_{i,f}x}$

- Inserting the free particle solutions leads to

$$T_{fi} = -i \int d^4x \, A^\mu (-ie) \left[\phi_f^* \partial_\mu \phi_i - \partial_\mu \phi_f^* \phi_i \right]$$

$$= -i \int d^4x \, A^\mu(x) \underbrace{(-)eN_i N_f \left(p_i + p_f \right)_\mu e^{-i(p_i - p_f)x}}_{J_\mu^{(fi)}}$$

$J_\mu^{(fi)}$

ϕ_i ϕ_f

A^μ

2- Matrix element

- Consider the process : $a+1 \rightarrow b+2$

$J_\mu^{(ab)}$

p_a p_b

A^μ

p_1 p_2

$J_\mu^{(12)}$

The intermediate field is no more a free particle field but is induced by the motion of the bottom part of the diagram.

The A^μ field is given by the photon propagation equations in presence of a source :

$$\left(\partial^\nu \partial_\nu\right) A^\mu(x) - \partial^\mu \underbrace{\left(\partial^\nu A_\nu\right)}_{\substack{=0 \\ \text{in Lorentz} \\ \text{condition}}} = J^\mu_{(12)}$$

- The current is given by :

$$J_\mu^{(12)} = \left\langle p_2 \middle| \hat{J}_\mu \middle| p_1 \right\rangle = -e N_1 N_2 \left(p_1 + p_2\right)_\mu e^{-i(p_1 - p_2).x}$$

2- Matrix element

- The previous expression allows to compute the A^μ field :

$$\Box A_\mu = J_\mu^{(12)} = -e N_1 N_2 \left(p_1 + p_2 \right)_\mu e^{-i(p_1 - p_2).x} = -e N_1 N_2 \left(p_1 + p_2 \right)_\mu e^{iq.x}$$

$$\Rightarrow A_\mu = \left\langle 0 \middle| \hat{A}_\mu \middle| q \right\rangle = +\frac{1}{q^2} e N_1 N_2 \left(p_1 + p_2 \right)_\mu e^{iq.x}$$

$$\Rightarrow A_\mu = \frac{-g_\mu^{\ \nu}}{q^2} J_\nu^{(12)}$$

- Replacing this expression in the matrix element :

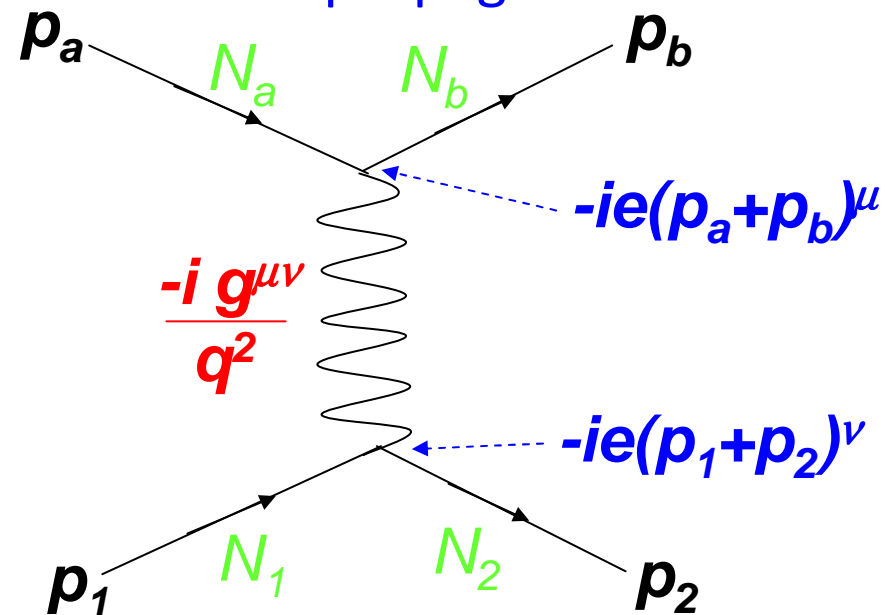$$T_{fi} = -i \int d^4 x\, J_\mu^{(ab)} \frac{-g^{\mu\nu}}{q^2} J_\nu^{(12)}$$

- Namely :

$$T_{fi} = -i(-e)(-e) N_a N_b N_1 N_2 \left(p_a + p_b \right)_\mu \frac{-g^{\mu\nu}}{q^2} \left(p_1 + p_2 \right)_\nu$$

$$\times (2\pi)^4 \delta^4 \left(p_f - p_i - k \right)$$

Summary : "reading" a Feynman diagram

- The basic elements of a diagram are :

  - the external lines

  - the vertex operator

  - the internal lines or propagators

with the transferred momentum : $q = \left(p_a - p_b \right) = \left(p_2 - p_1 \right)$

3- Feynman rules for QED

Including fermions into the game: same procedure

$\Rightarrow$ matrix element reduced expression

$$T_{fi} = -i \int d^4x \, J_\mu^{(ab)} \frac{-g^{\mu\nu}}{q^2} J_\nu^{(12)}$$

$\Rightarrow$ with a current given by (see lecture on Dirac's equation) :

$$J_\mu^{(fi)} = -e\left(\bar{\psi}_f \gamma_\mu \psi_i\right) = -e\left(\bar{u}_f \gamma_\mu u_i\right) e^{i(p_f - p_i).x}$$

$\Rightarrow$ N.B. to be compared with the spinless case :

$$J_\mu^{(fi)} = -e\left(p_f + p_i\right)_\mu e^{i(p_f - p_i).x}$$

Feynman rules for QED:

- **External lines:**

  - fermions =>   *(incoming)* $\longrightarrow$ ◯ $u(p,s)$ ◯ *(outgoing)* $\longrightarrow$ $\overline{u}(p',s')$

  - anti-fermions => $\longrightarrow$ ◯ $\overline{v}(p,s)$ ◯ $\longrightarrow$ $v(p',s')$

  - spin-0 bosons => $----\!\!\rightarrow---$ ◯ cst ◯ $----\!\!\rightarrow---$ cst

  - photons => $\sim\!\!\sim\!\!\rightarrow\!\!\sim\!\!\sim$ ◯ $\varepsilon_{\mu}(k,\lambda)$ ◯ $\sim\!\!\sim\!\!\rightarrow\!\!\sim\!\!\sim$ $\varepsilon_{\mu}^{*}(k',\lambda')$

3- Feynman rules for QED

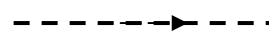Feynman rules for QED:

- **Propagators:**
  - fermions => $$\frac{i}{p\!\!\!/ - m} = \frac{i(p\!\!\!/ + m)}{p^2 - m^2}$$

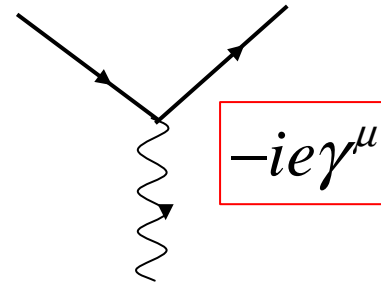 - spin-0 bosons => $$\frac{i}{p^2 - m^2}$$

  - photons => $$\frac{-ig^{\mu\nu}}{p^2}$$

3- Feynman rules for QED

Feynman rules for QED:

- • Vertices :

  - fermions – photon =>

  $$-ie\gamma^{\mu}$$

  - spin-0 bosons – photon =>

  $$-ie(p + p')^{\mu}$$