Chapter 3

Relativistic propagation equation for bosons

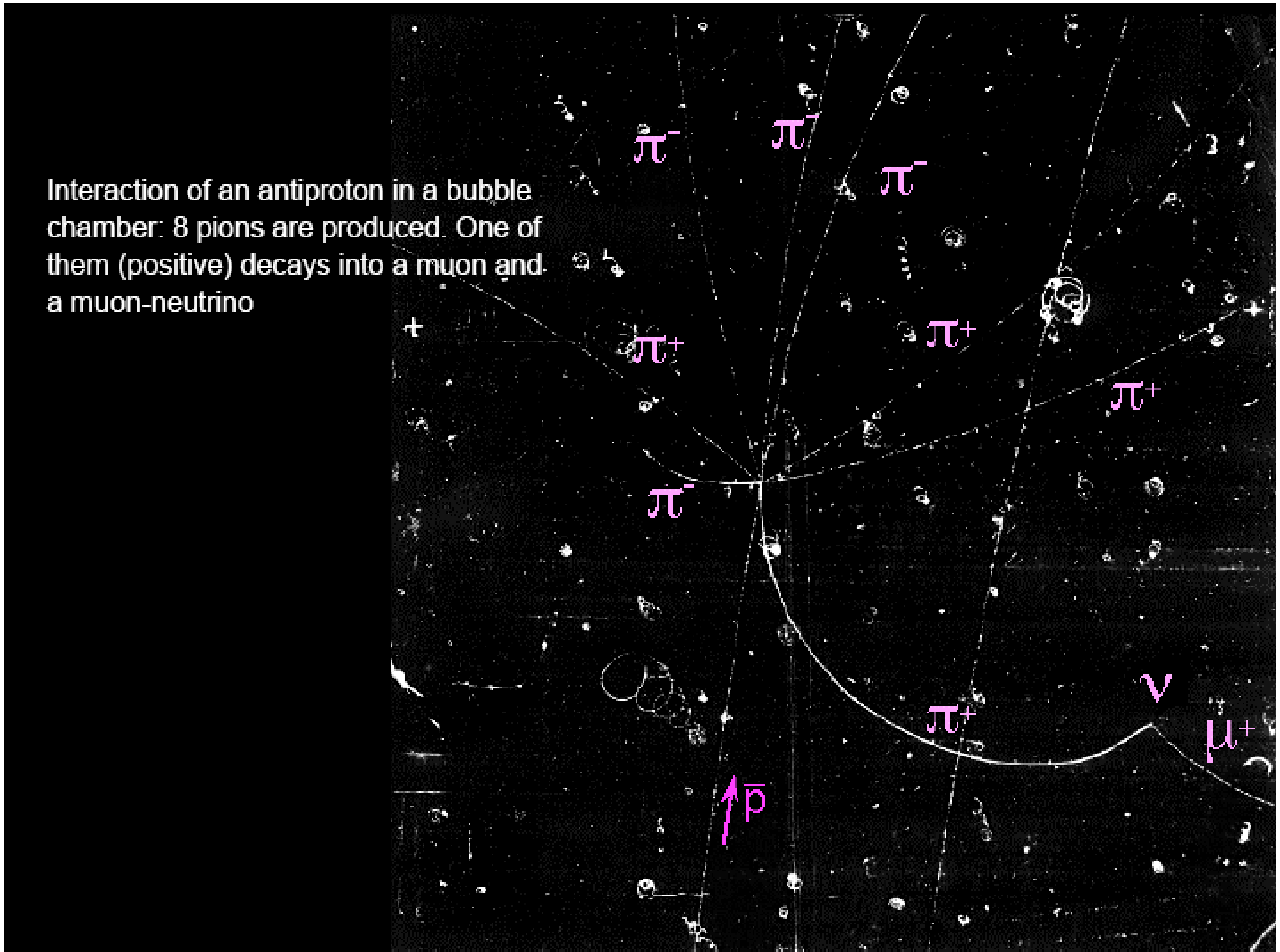
Outline/Plan

1. **Antiparticles and relativity**
2. **Canonical quantization**
3. **Klein-Gordon equation**
    1. KG equation derivation
    2. Probabilistic interpretation
    3. Diffusion amplitude

1. **Antiparticules et relativité**
2. **Quantification canonique**
3. **Equation de Klein-Gordon**
    1. Dérivation de l'équation de KG
    2. Interprétation probabiliste
    3. Amplitude de diffusion

1- Antiparticles and relativity

Observation of antiparticles :

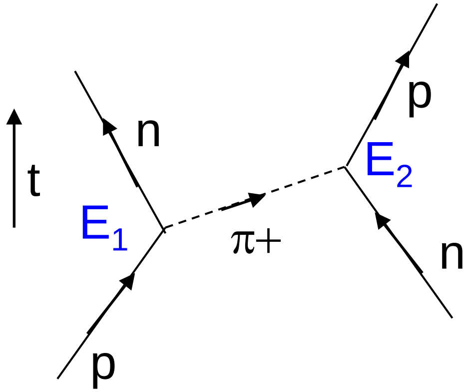


1- Antiparticles and relativity

Historically:

- Predication of the positron by Dirac (1928)
- Experimental signature by Anderson (1932)
- Theoretical difficulties : “negative energies” ➔ holes theory
- Matter-antimatter asymmetry

Studying charge-exchange diagram with some basics of 4-vectors:



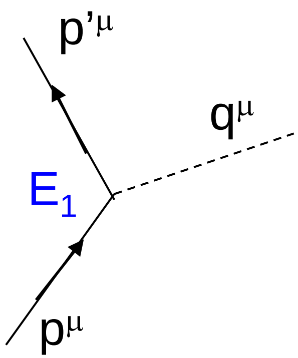
$$p^\mu = (p^0 = E = \gamma M, \vec{p} = \gamma M \vec{v})$$

$$\gamma = 1 / \sqrt{1-\beta^2} \quad \text{and} \quad \beta = v / c$$

$$p^2 = p^\mu p_\mu = E^2 - \vec{p}^2 = E^2(1 - v^2) = M^2$$

1- Antiparticles and relativity

The charge exchange process is forbidden classically



$$p^{\mu} = p'^{\mu} + q^{\mu}$$

$$q^2 = (p^{\mu} - p'^{\mu})^2 = 2M^2 - 2p^{\mu}p'_{\mu}$$

$$= 2M^2\left(1 - \frac{1-\vec{v}.\vec{v}'}{\sqrt{(1-v^2)(1-v'^2)}}\right)$$

$$\leq 0 \text{ but } q^2 = m^2\,?$$

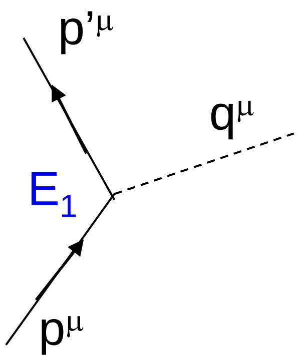
- Between E_1 and E_2 a virtual particle state may be exchanged for a time less than $\Delta t \leq \frac{\hbar}{m}$
- "E1 – E2" : space-like interval between the 2 events

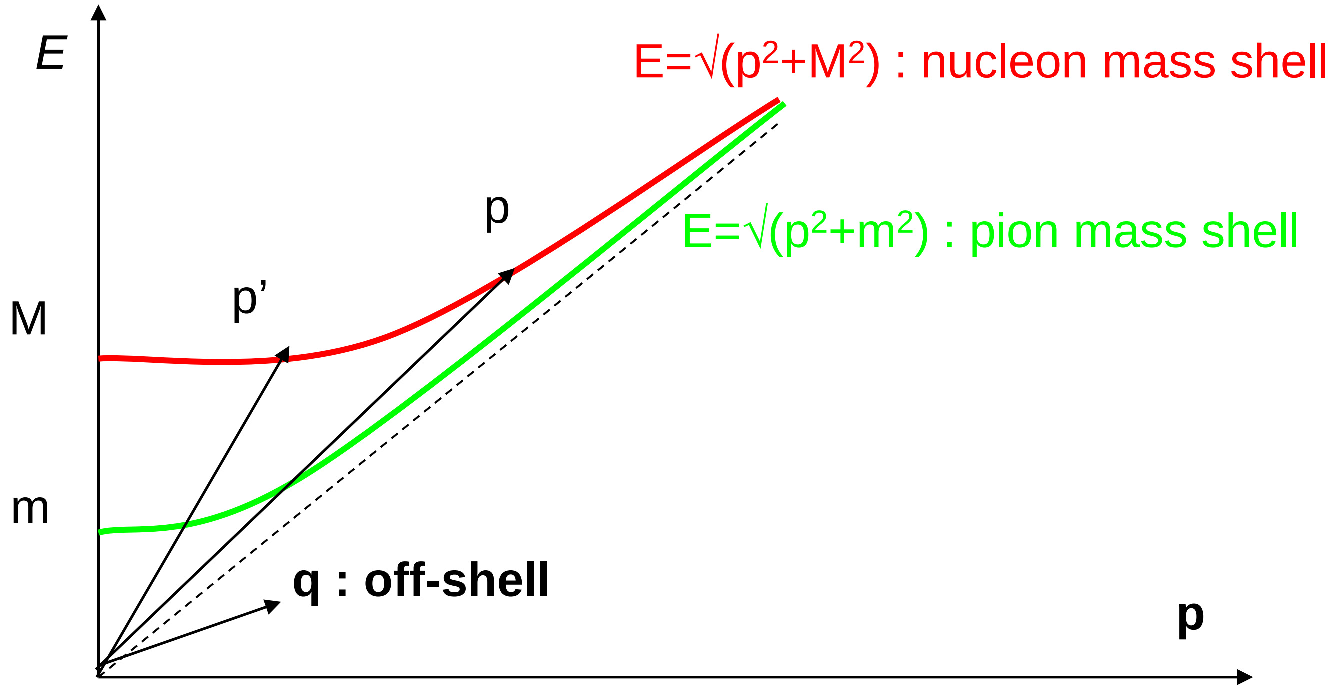
$$q^2 = q_0^2 - \vec{q}^{\,2} = q_0^2\left(1 - V^2\right) \text{ with } V: \text{ pion velocity}$$

$$\Rightarrow (\Delta t)^2 - (\Delta\vec{x})^2 = (\Delta t)^2\left(1 - V^2\right) < 0$$

1- Antiparticles and relativity

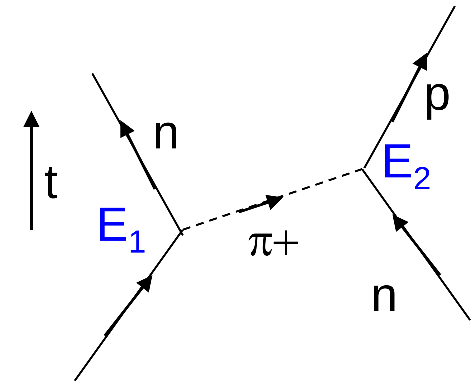
Off-shell virtual particle exchange :



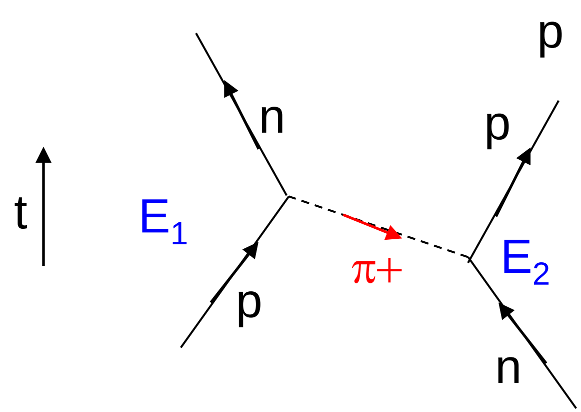


1- Antiparticles and relativity

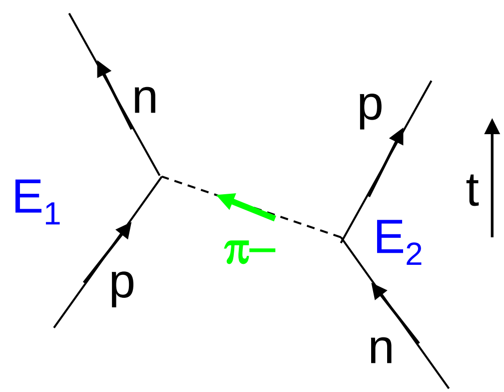
By reversing time order (E2 observed before E1) the π^+ should have been absorbed before emission (causality violation)



E1 before E2



E1 after E2



$\Rightarrow$emission of π^+ antiparticle : π^-

2- Canonical quantization

Introduction :

- Schrödinger equation describes the evolution of a non-relativistic wavefunction using the canonical quantization :

$$\vec{\mathrm{x}} \rightarrow \vec{\mathrm{x}}$$

$$\vec{p} \rightarrow -i\hbar\vec{\nabla}$$

$$E \rightarrow i\hbar\tfrac{\partial}{\partial t}$$

applied to the energy definition :

$$E = \frac{\vec{p}^{\,2}}{2m} + V(\vec{x})$$

$$\Rightarrow i\hbar\frac{\partial\psi}{\partial t}(\vec{x},t) = \left(\frac{-\hbar^2}{2m}\Delta + V\right)\psi(\vec{x},t)$$

2- Canonical quantization

4D covariant generalization:

- Reminder : covariant formalism

$$x^{\mu} \equiv \left(x^{0} = ct, \vec{x}\right) \text{ and } x_{\mu} \equiv \left(x^{0} = ct, -\vec{x}\right)$$

$$\partial_{\mu} \equiv \frac{\partial}{\partial x^{\mu}} = \left(\frac{\partial}{c\partial t}, \vec{\nabla}\right) \text{ and } \partial^{\mu} \equiv \frac{\partial}{\partial x_{\mu}} = \left(\frac{\partial}{c\partial t}, -\vec{\nabla}\right)$$

- 4-dimensional canonical quantization :

$$\boxed{p^{\mu} \equiv \left(p^{0} = E, \vec{p}\right) \rightarrow i\hbar\partial^{\mu} = \left(i\hbar\frac{\partial}{c\partial t}, -i\hbar\vec{\nabla}\right)}$$

2- Canonical quantization

Q : how to derive a relativistic evolution equation?

A : Following the same prescription than in the Schrödinger case but with the relativistic energy definition :

$$E^2 = \vec{p}^2 c^2 + m^2 c^4$$

3-1 K-G equation derivation

Preliminary remark : why not starting from

$$E = \sqrt{\vec{p}^2 c^2 + m^2 c^4} \qquad ?$$

- Time-space asymmetry
- Difficult development of the square root
- But : only positive-energy solutions…

3-1 K-G equation derivation

Derivation : canonical quantization applied

$$p^2 = p^\mu p_\mu = p_0^2 - \vec{p}^2 = m^2c^2$$

$$\Downarrow$$

$$-\hbar^2 \partial^\mu \partial_\mu \psi(x) = m^2c^2\psi(x) \text{ with } \partial^\mu \partial_\mu \equiv \Box = \frac{\partial^2}{c^2 \partial t^2} - \Delta$$

Finally :

$$\left(\Box + \frac{m^2c^2}{\hbar^2} \right) \psi(x) = 0$$

Klein-Gordon equation

3-1 K-G equation derivation

Remarks (I) :

- The photon (boson) is solution of KG's propagation equation (with m=0)
- KG allows to describe all (anti-)particles within the same formalism

3-1 K-G equation derivation

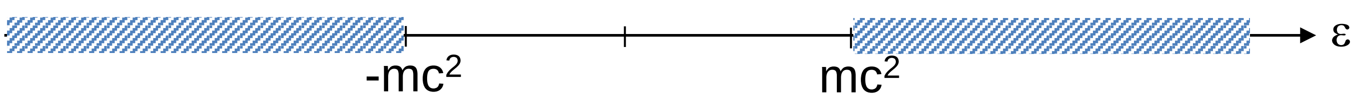
Remarks (II) :

- Negative energy solutions? Consider a plane wave

$$\psi(x) = Be^{i(\vec{p}.\vec{x}-\varepsilon t)/\hbar} = Be^{ip^{\mu}x_{\mu}/\hbar}$$

$$\left(\Box + \frac{m^2c^2}{\hbar^2}\right)\psi = 0 \Rightarrow \varepsilon^2 = \vec{p}^2c^2 + m^2c^4$$

$$\varepsilon = \pm E_p \text{ with } E_p = \sqrt{\vec{p}^2c^2 + m^2c^4}$$



- Interpretation of that spectrum?

3-2 Probabilistic interpretation

Reminder :

- In the non-relativistic case the probabilistic interpretation of wavefunctions reads (continuity equation) :

$$\frac{\partial \rho}{\partial t} + \vec{\nabla}.\vec{J} = 0 \text{ where}$$

$$\begin{cases} \rho = \psi \psi^{*} \\ \vec{J} = \Re\left(\dfrac{-i\hbar}{m} \psi^{*} \vec{\nabla} \psi \right) = \dfrac{-i\hbar}{2m} \left(\psi^{*} \vec{\nabla} \psi - \psi \vec{\nabla} \psi^{*} \right) \end{cases}$$

with the integral normalization condition :

$$\int d^{3}x \left| \psi(\vec{x}, t) \right|^{2} = N = ct$$

3-2 Probabilistic interpretation

In the relativistic case the same procedure can be followed :

$$\partial^{\mu}\partial_{\mu}\psi(x)+m^2c^2\psi(x)\ =0$$

$$\Rightarrow \begin{cases} \psi^*\partial^{\mu}\partial_{\mu}\psi(x)+m^2c^2\psi^*\psi(x)\ =0 \\ \psi\partial^{\mu}\partial_{\mu}\psi^*(x)+m^2c^2\psi\psi^*(x)\ =0 \end{cases}$$

$$\Rightarrow \psi^*\partial^{\mu}\partial_{\mu}\psi-\psi\partial^{\mu}\partial_{\mu}\psi^*=0$$

$$\Leftrightarrow \partial^{\mu}J_{\mu}=0 \text{ with } J_{\mu}=\frac{i\hbar}{2m}\left(\psi^*\partial^{\mu}\psi-\psi\partial^{\mu}\psi^*\right)$$

3-2 Probabilistic interpretation

Integral charge conservation condition :

$$\int d^3x \rho(x) = N = ct$$

where

$$\rho(x) = \frac{i\hbar}{2mc^2}\left(\psi^* \frac{\partial}{\partial t}\psi - \psi \frac{\partial}{\partial t}\psi^*\right)$$

and $\rho \leq \text{ or } \geq 0$

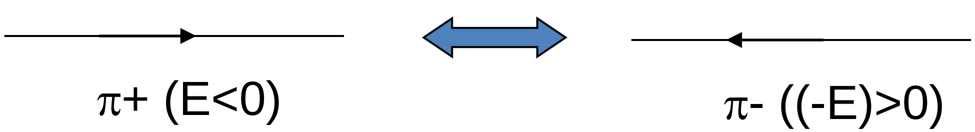
3-2 Probabilistic interpretation

Interpretation? The number of particles is not conserved (possible annihilations) but some charges may be conserved : multiplying ρ by the electric charge in the case of a plane wave for instance:

$$\psi_\pm(x) = B_\pm e^{i(\vec{p}.\vec{x} \mp E_p t)/\hbar} \Rightarrow \rho_\pm = \pm e \frac{E_p}{mc^2} \left| B_\pm \right|^2$$

$\rho_\pm$ represents the charge density which may be of both signs!

A negative energy particle represents an anti-particle moving in reverse time order!



3-3 Diffusion amplitude

Reminder : transition amplitude (covariant expression) from initial (i) to final *(f)* state on the action of a perturbation potential *V.*

$$T_{fi} = -i\int d^4x\, \psi_f^*(x)V(x)\psi_i(x)$$

For a charged particle moving in an (e.g. electromagnetic) potential A^μ : $p^\mu \rightarrow p^\mu + eA^\mu$ ie $i\hbar\partial^\mu \rightarrow i\hbar\partial^\mu + eA^\mu$

The KG equation reads therefore :

$$\left(\partial^\mu\partial_\mu + m^2\right)\psi(x) = -V\psi(x) \text{ where } V = -ie\left(\partial_\mu A^\mu + A^\mu\partial_\mu\right) + o(e^2)$$

3-3 Diffusion amplitude

Amplitude computation :

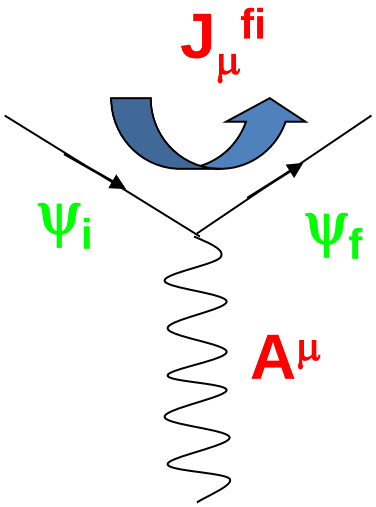
$$T_{fi} = -i\int d^4x\, \psi_f^*(x)V(x)\psi_i(x)$$

$$T_{fi} = -i\int d^4x\, \psi_f^*(x)\Big(-ie\big(\partial_\mu A^\mu + A^\mu\partial_\mu\big)\Big)\psi_i(x)$$

$$= -i\int d^4x\,(-ie)\Big[\psi_f^* A^\mu\partial_\mu\psi_i - \partial_\mu\psi_f^* A^\mu\psi_i\Big]$$

$$= -i\int d^4x\, A^\mu(-ie)\Big[\psi_f^*\partial_\mu\psi_i - \partial_\mu\psi_f^*\psi_i\Big]$$

$$= -i\int d^4x\, A_\mu J^\mu_{fi}$$



With A^μ linked to its source through $\big(\partial^\nu\partial_\nu\big)A^\mu(x) = J^\mu_{(2)}$

(see later)

3-3 Diffusion amplitude

Starting with Feynman diagrams :

$$T_{fi} = -i\int d^4x\, A_\mu J^\mu_{(1)} = -i\int d^4x\, J_{\mu(2)}\left(\frac{-1}{q^2}\right) J^\mu_{(1)}$$