Dirac equation: spinors, γ matrices, helicity, chirality

1 / Write down a simplified form of the Dirac equation for a spinor $\Psi(t)$ describing a particle of mass m at rest. For the standard Pauli-Dirac representation of the γ matrices, obtain a differential equation for each component ψ_i of the spinor Ψ, and hence write down a general solution for the evolution of Ψ.

2 / For the standard Pauli-Dirac representation of the γ matrices, and for an arbitrary pair of spinors Ψ and Φ with components ψ_i et ϕ_i

a- Calculate the components of the quadricurrent $\overline{\Psi}\gamma^\mu\Phi$ (as a function of ψ_i et ϕ_i).

b- For a particle or antiparticle with four-momentum $p^\mu = (E, p_x, p_y, p_z)$, show that

$$\overline{u_1}\gamma^\mu u_1 = \overline{u_2}\gamma^\mu u_2 = \overline{v_1}\gamma^\mu v_1 = \overline{v_2}\gamma^\mu v_2 = 2p^\mu$$

and that

$$\overline{u_1}\gamma^\mu u_2 = \overline{u_2}\gamma^\mu u_1 = \overline{v_1}\gamma^\mu v_2 = \overline{v_2}\gamma^\mu v_1 = 0$$

c- Hence show that the current $j^\mu = \overline{\Psi}\gamma^\mu\Phi$ corresponding to a general free particle spinor $\Psi(p) = u(p)e^{i(\vec{p}.\vec{r}-Et)}$ or antiparticle spinor $\Psi(p) = v(p)e^{-i(\vec{p}.\vec{r}-Et)}$ is given by $j^\mu = 2p^\mu$. Write down the particle density and flux represented by j^μ, and show that they are consistent.

3 / Calculate, in the helicity basis, the positive energy spinor for an electron with momentum $\vec{p} = (p\sin\theta, 0, p\cos\theta)$.

4 / Show that $\Lambda_+ = \frac{\not{p}+m}{2m}$ and $\Lambda_- = \frac{-\not{p}+m}{2m}$ are the projectors on positive and negative energy state.

5 / Show the completeness relations :

$$\sum_s u(p,s)\overline{u}(p,s) = \not{p} + m$$

$$\sum_s v(p,s)\overline{v}(p,s) = \not{p} - m$$

6 / Show that the chirality eigenstates are identical to the helicity eigenstates for a massless spin 1/2 particle with energy E

7 / Show that $\overline{\Psi}_L\gamma^\mu\Psi_R = \overline{\Psi}_R\gamma^\mu\Psi_L = 0$ and that the current $j^\mu = \overline{\Psi}\gamma^\mu\Psi$ can be written

$$\overline{\Psi}\gamma^\mu\Psi = \overline{\Psi_L}\gamma^\mu\Psi_L + \overline{\Psi_R}\gamma^\mu\Psi_R$$

8 / In the framework of the Dirac equation, show that:

a- $[H, \vec{L}] = -i\,\vec{\alpha}\times\vec{p}$

b- $[H, \vec{\Sigma}] = 2i\,\vec{\alpha}\times\vec{p}$

c- What about $\vec{L} + \vec{\Sigma}/2$? conclusions ?