Chapter 7
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Basic electromagnetic processes

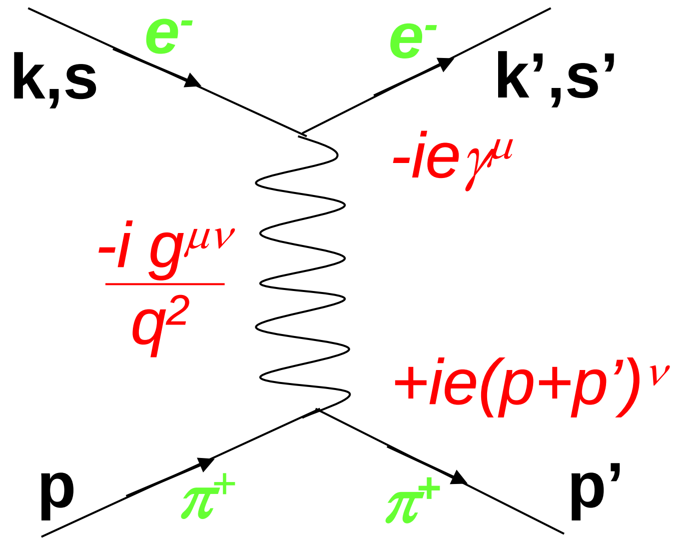
Outline/Plan

1. Electron-pion scattering
    1. Transition amplitude
    2. Trace techniques for spin summations
    3. cross-section
2. Electron-muon scattering

1. Diffusion électron-pion
    1. Amplitude de transition
    2. Techniques de traces pour la sommation sur les spins
    3. Section efficace
2. Diffusion électron-muon

1- Electron-pion scattering

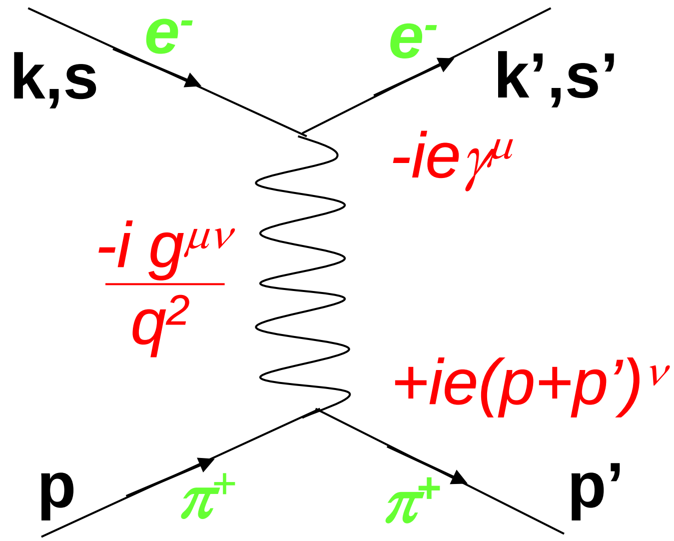
Applications of the Feynman rules to the $e^- \pi^+$ elastic scattering



$$d\sigma = \frac{1}{2E_k 2E_p \left|\vec{v}_k - \vec{v}_p\right|} \int (2\pi)^4 \delta^4(k' + p' - k - p) \overline{\left|T_{fi}\right|^2} \frac{d^3k'}{(2\pi)^3 2E_{k'}} \frac{d^3p'}{(2\pi)^3 2E_{p'}}$$

1-1 Transition amplitude

Expression of the transition amplitude :



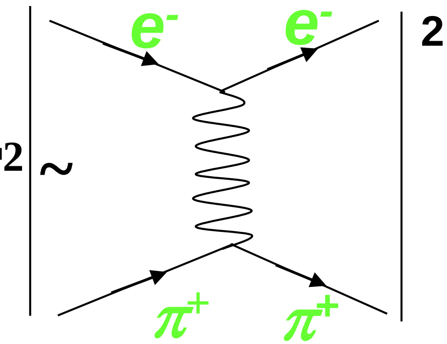
$$T = \overline{u}(k',s')(-ie\gamma^{\mu})u(k,s)\frac{-ig_{\mu\nu}}{q^2}(+ie)(p+p')^{\nu}$$

$$= \frac{-ie^2}{q^2}\overline{u}(k',s')\gamma^{\mu}u(k,s)(p+p')_{\mu}$$

1-1 Transition amplitude

Expression of the |transition amplitude|^2 :

$\mathrm{T}^2 \sim$



$$T^2 = \left(\frac{e^2}{q^2}\right)^2 \left[\overline{u}(k',s')\gamma^\nu u(k,s)(p+p')_\nu\right]\left[\overline{u}(k',s')\gamma^\mu u(k,s)(p+p')_\mu\right]^*$$

with the conjugate expression reading :

$$\left[\overline{u}(k',s')\gamma^\mu u(k,s)(p+p')_\mu\right]^* = \left[\overline{u}(k',s')\gamma^\mu u(k,s)\right]^\dagger (p+p')_\mu$$

$$\begin{aligned}\left[\overline{u}(k',s')\gamma^\mu u(k,s)\right]^\dagger &= u(k,s)^\dagger \gamma^{\mu\dagger}(u^\dagger\gamma^0)(k',s')^\dagger \\ &= u(k,s)^\dagger \gamma^0\gamma^\mu\gamma^0\gamma^0 u(k',s') \\ &= \overline{u}(k,s)\gamma^\mu u(k',s')\end{aligned}$$

1-2 Trace techniques for spin $\sum$

Warning : most of the time the simplest measurements use unpolarized electrons (beam of spin-↑ and spin-↓ electrons) and do not include the information on the polarization.

The unpolarized cross-section is defined by :

$$\boxed{d\bar{\sigma} \equiv \frac{1}{2s+1}\left(d\sigma_{\uparrow\uparrow} + d\sigma_{\uparrow\downarrow} + d\sigma_{\downarrow\uparrow} + d\sigma_{\downarrow\downarrow}\right) = \frac{1}{2}\sum_{s}\sum_{s'} d\sigma_{ss'}}$$

This apparent extra complexity leads to an over simplification of the computation thanks to the 'trace technique'.

1-2 Trace techniques for spin $\sum$

Expression of the spin-averaged |transition amplitude|^2 :

$$\left|T^2\right| = \left(\frac{e^2}{q^2}\right)^2 \frac{1}{2} \sum_s \sum_{s'} \left[\overline{u}(k',s')\gamma^\mu u(k,s)\right]\left[\overline{u}(k,s)\gamma^\nu u(k',s')\right]$$

$$\times (p+p')_\mu (p+p')_\nu$$

$$\equiv \left(\frac{e^2}{q^2}\right)^2 L^{\mu\nu} P_{\mu\nu}$$

Definition of leptonic and hadronic (here pionic) tensors.

The hadronic tensor is simply : $P_{\mu\nu} = (p+p')_\mu (p+p')_\nu$

1-2 Trace techniques for spin $\sum$

The complete expression of the leptonic tensor reads :

$$L^{\mu\nu} = \frac{1}{2}\sum_{ss'}\sum_{\alpha\beta\delta\varepsilon}\bar{u}_\alpha(k',s')(\gamma^\mu)_{\alpha\beta}u_\beta(k,s)\bar{u}_\delta(k,s)(\gamma^\nu)_{\delta\varepsilon}u_\varepsilon(k',s')$$

$$= \frac{1}{2}\sum_{\alpha\beta\delta\varepsilon}\underbrace{\sum_{s'}u_\varepsilon(k',s')\bar{u}_\alpha(k',s')}_{(\not{k}'+m_e)_{\varepsilon\alpha}}(\gamma^\mu)_{\alpha\beta}\underbrace{\sum_{s}u_\beta(k,s)\bar{u}_\delta(k,s)}_{(\not{k}+m_e)_{\beta\delta}}(\gamma^\nu)_{\delta\varepsilon}$$

$$= \frac{1}{2}\sum_{\alpha\beta\delta\varepsilon}(\not{k}'+m_e)_{\varepsilon\alpha}(\gamma^\mu)_{\alpha\beta}(\not{k}+m_e)_{\beta\delta}(\gamma^\nu)_{\delta\varepsilon}$$

$$= \frac{1}{2}\mathrm{Tr}\left[(\not{k}'+m_e)(\gamma^\mu)(\not{k}+m_e)(\gamma^\nu)\right]$$

1-2 Trace techniques for spin $\sum$

Trace theorems :

$$\mathrm{Tr}\left[1_{4\times4}\right]=4$$

$$\mathrm{Tr}\left[\text{nbr impair de }\gamma\right]=0$$

$$\mathrm{Tr}\left[\gamma^{\mu}\gamma^{\nu}\right]=4\left(g^{\mu\nu}\right)$$

$$\mathrm{Tr}\left[\gamma^{\alpha}\gamma^{\mu}\gamma^{\beta}\gamma^{\nu}\right]=4\left(g^{\alpha\mu}g^{\beta\nu}-g^{\alpha\beta}g^{\mu\nu}+g^{\alpha\nu}g^{\beta\mu}\right)$$

$$\mathrm{Tr}\left[\gamma^{5}\right]=0$$

$$\mathrm{Tr}\left[\gamma^{5}\gamma^{\alpha}\gamma^{\mu}\right]=0$$

$$\mathrm{Tr}\left[\gamma^{5}\gamma^{\alpha}\gamma^{\mu}\gamma^{\beta}\gamma^{\nu}\right]=4i\varepsilon_{\eta\rho\sigma\tau}g^{\eta\alpha}g^{\rho\mu}g^{\sigma\beta}g^{\tau\nu}$$

1-3 Cross-section

Leptonic tensor (cont'd) :

$$L^{\mu\nu} = \frac{1}{2}\mathrm{Tr}\left[(\not{k}'+m_e)(\gamma^\mu)(\not{k}+m_e)(\gamma^\nu)\right]$$

$$= k'_\alpha k_\beta \frac{1}{2}\mathrm{Tr}\left[\gamma^\alpha\gamma^\mu\gamma^\beta\gamma^\nu\right] + m_e^2\frac{1}{2}\mathrm{Tr}\left[\gamma^\mu\gamma^\nu\right]$$

$$= \frac{1}{2}4k'_\alpha k_\beta\left(g^{\alpha\mu}g^{\beta\nu} - g^{\alpha\beta}g^{\mu\nu} + g^{\alpha\nu}g^{\beta\mu}\right) + m_e^2\frac{1}{2}4g^{\mu\nu}$$

$$= 2\left(k'^\mu k^\nu - (k'.k)g^{\mu\nu} + k^\mu k'^\nu\right) + 2m_e^2 g^{\mu\nu}$$

Transferred momentum : $q = k - k' = p' - p$

$$\Rightarrow q^2 = k^2 + k'^2 - 2k.k' = 2(m^2 - k.k')$$

Therefore : $L^{\mu\nu} = \left[2\left(k'^\mu k^\nu + k^\mu k'^\nu\right) + q^2 g^{\mu\nu}\right]$

1-3 Cross-section

Finally : $$L^{\mu\nu}P_{\mu\nu} = \left[2\left(k'^{\mu}k^{\nu} + k^{\mu}k'^{\nu}\right) + q^2 g^{\mu\nu}\right](p+p')_{\mu}(p+p')_{\nu}$$

$$\text{with } q_{\mu}L^{\mu\nu} = q_{\nu}L^{\mu\nu} = 0$$

$$L^{\mu\nu}P_{\mu\nu} = \left[2\left(k'^{\mu}k^{\nu} + k^{\mu}k'^{\nu}\right) + q^2 g^{\mu\nu}\right](2p)_{\mu}(2p)_{\nu}$$

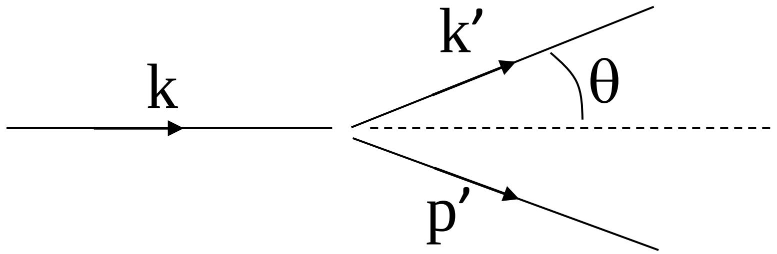
$$= 8\left[2(k'.p)(k.p)\right] + 4m_{\pi}^2 q^2$$

And the cross-section reads :

$$d\sigma = \frac{1}{2E_k 2E_p \left|\vec{v}_k - \vec{v}_p\right|}\left(\frac{4\pi\alpha}{q^2}\right)^2 \left(8\left[2(k'.p)(k.p)\right] + 4m_{\pi}^2 q^2\right) d\Phi_2$$

In the 'laboratory' frame

(pion at rest initially) :



1-3 Cross-section

The phase space is given by :

$$d\Phi_2 = \frac{1}{16\pi^2}\frac{\left|\vec{k}\,'\right|^3 d\Omega}{\left|\vec{k}\,'\right|^2 E_T - \vec{k}\,'.\vec{k}E_{k'}} \quad \text{with } E_T = m_\pi + E_k,\ E_k \approx \left|\vec{k}\right|,\ E_{k'} \approx \left|\vec{k}\,'\right|$$

$$= \frac{1}{16\pi^2}\frac{E_{k'}d\Omega}{m_\pi + E_k - E_k\cos\theta}$$

$$= \frac{1}{16\pi^2}\frac{E_{k'}}{m_\pi}\frac{d\Omega}{1+2\sin^2(\theta/2)\dfrac{E_k}{m_\pi}}$$

$$\frac{\left|\vec{k}\right|}{\left|\vec{k}\,'\right|} = \frac{E_k}{E_{k'}}$$ **Elastic scattering condition**

$$= \frac{1}{16\pi^2}\frac{E_{k'}^2}{m_\pi E_k}$$

where the electron mass has been neglected

$$\Rightarrow q^2 = 2(m^2 - k.k') \approx -4E_kE_{k'}\sin^2\theta/2$$

1-3 Cross-section

The cross-section final expression is :

$$\frac{d\sigma}{d\Omega} = \frac{1}{4m_\pi E_k}\left(\frac{4\pi\alpha}{-4E_{k'}E_k \sin^2\theta/2}\right)^2 16(E_{k'}E_k)m_\pi^2\left(1-\sin^2\theta/2\right)$$

$$\times\frac{1}{16\pi^2}\frac{E_{k'}^2}{m_\pi E_k}$$

$$\boxed{\frac{d\sigma}{d\Omega} = \frac{\alpha^2}{4E_k^2\sin^4\theta/2}\left(\cos^2\theta/2\right)\frac{E_{k'}}{E_k} \equiv \left(\frac{d\sigma}{d\Omega}\right)_{ns}}$$

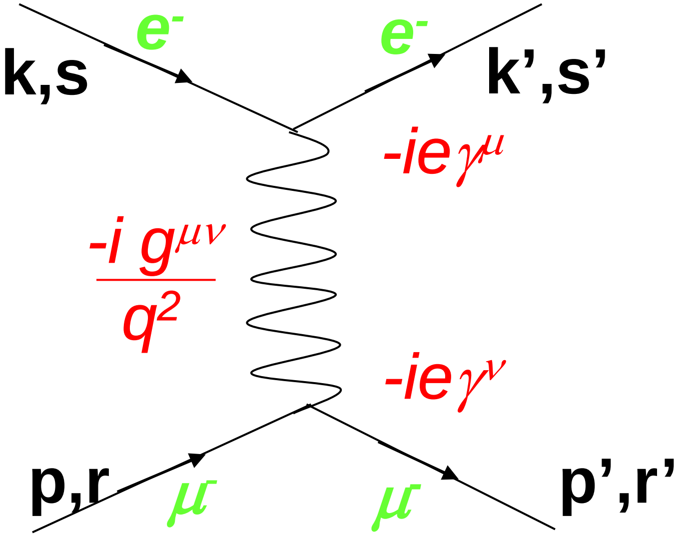
N.B. for a real pion one has to take into account the form factor :

$$-ie(p+p')^\mu \rightarrow -ieF(q^2)(p+p')^\mu$$

the form of the e.m. current is deduced from the Lorentz invariance requirements

2- e-μ scattering

Consider now a process involving 4 fermions in the initial-final states (i.e. 4 spin indexes) :



The computation procedure is identical to what was done before :

- $T = \overline{u}(k',s')(-ie\gamma^\mu)u(k,s)\dfrac{-ig_{\mu\nu}}{q^2}\overline{u}(p',r')(-ie\gamma^\nu)u(p,r)$

- $\left|\overline{T}\right|^2 = \dfrac{1}{(2s_e+1)(2s_\mu+1)}\sum_{rr'ss'} TT^* = \dfrac{1}{4}\sum_{rr'ss'} TT^*$

2- e-μ scattering

Expression of the average transition amplitude in terms of tensor product:

$$|T|^2 = \left(\frac{e^2}{q^2}\right)^2 \frac{1}{4}\sum_{rr'ss'}\left[\overline{u}(k',s')\gamma^{\mu}u(k,s)\overline{u}(p',r')\gamma_{\mu}u(p,r)\right]$$

$$\times\left[\overline{u}(k,s)\gamma^{\nu}u(k',s')\overline{u}(p,r)\gamma_{\nu}u(p',r')\right]$$

$$= \left(\frac{e^2}{q^2}\right)^2 \frac{1}{2}\sum_{ss'}\left[\overline{u}(k',s')\gamma^{\mu}u(k,s)\overline{u}(k,s)\gamma^{\nu}u(k',s')\right]$$

$$\times\frac{1}{2}\sum_{rr'}\left[\overline{u}(p',r')\gamma_{\mu}u(p,r)\overline{u}(p,r)\gamma_{\nu}u(p',r')\right]$$

$$= \left(\frac{e^2}{q^2}\right)^2 \frac{1}{2}Tr\left[(\not{k}'+m_e)\gamma^{\mu}(\not{k}+m_e)\gamma^{\nu}\right]\times\frac{1}{2}Tr\left[(\not{p}'+m_{\mu})\gamma_{\mu}(\not{p}+m_{\mu})\gamma_{\nu}\right]$$

$$= \left(\frac{e^2}{q^2}\right)^2 L^{\mu\nu}(e)L_{\mu\nu}(\mu)$$

2- e-μ scattering

The leptonic tensors have the form computed previously :

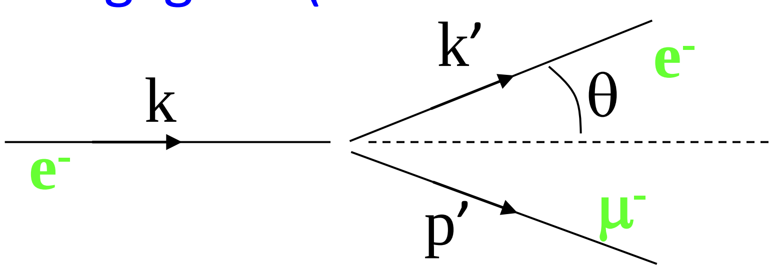
$$L^{\mu\nu}(e) = \frac{1}{2}\mathrm{Tr}\left[(\not{k}'+m_e)(\gamma^\mu)(\not{k}+m_e)(\gamma^\nu)\right]$$

$$= 2\left(k'^\mu k^\nu - (k'.k)g^{\mu\nu} + k^\mu k'^\nu\right) + 2m_e^2 g^{\mu\nu}$$

$$= 2\left(k'^\mu k^\nu + k^\mu k'^\nu + (q^2/2)g^{\mu\nu}\right)$$

$$L^{\mu\nu}(\mu) = 2\left(p'^\mu p^\nu + p^\mu p'^\nu + (q^2/2)g^{\mu\nu}\right)$$

The tensor product is expressed in the 'lab' frame defined with the initial muon at rest and with the further assumption that the electron mass is negligible (same conventions as before).



2- e-μ scattering

Preliminary remark :

$$q_\mu L^{\mu\nu} = q_\nu L^{\mu\nu} = 0 \Rightarrow L^{\mu\nu}(e) L_{\mu\nu}(\mu) = L^{\mu\nu}(e) L^{eff}_{\mu\nu}(\mu)$$

$$\text{where } L^{eff}_{\mu\nu}(\mu) = 2\left(2 p_\mu p_\nu + (q^2/2) g_{\mu\nu}\right)$$

Therefore :

$$L^{\mu\nu}(e) L_{\mu\nu}(\mu) = 4\left(k'^\mu k^\nu + k^\mu k'^\nu + (q^2/2) g^{\mu\nu}\right)\left(2 p_\mu p_\nu + (q^2/2) g_{\mu\nu}\right)$$

$$= 4(4(p.k)(p.k') + (q^2/2)(2(k.k') + 2p^2) + 4(q^2/2)^2)$$

$$= 4(4(p.k)(p.k') + (q^2/2)(-q^2 + 2p^2) + 4(q^2/2)^2)$$

$$= 4(4(p.k)(p.k') + q^2 p^2 + (q^2)^2/2)$$

Where we used : $q^2 \approx -2k.k' \approx -4E_k E_{k'} \sin^2\frac{\theta}{2}$

2- e-μ scattering

Therefore :

$$L^{\mu\nu}(e)L_{\mu\nu}(\mu) = 4(4(p.k)(p.k') + q^2 p^2 + (q^2)^2/2)$$

$$= 4\Big(4m_\mu^2 E_k E_{k'} + 4(-E_k E_{k'} \sin^2\tfrac{\theta}{2})m_\mu^2 - 2E_k E_{k'} \sin^2\tfrac{\theta}{2} q^2)$$

$$= 16m_\mu^2 E_k E_{k'}(1 - \sin^2\tfrac{\theta}{2} - \sin^2\tfrac{\theta}{2}\frac{q^2}{2m_\mu^2})$$

$$= 16m_\mu^2 E_k E_{k'}(\cos^2\tfrac{\theta}{2} - \sin^2\tfrac{\theta}{2}\frac{q^2}{2m_\mu^2})$$

And

$$\boxed{\frac{d\sigma}{d\Omega} = \underbrace{\frac{\alpha^2}{4E_k^2 \sin^4\tfrac{\theta}{2}}\left(\cos^2\tfrac{\theta}{2}\right)\frac{E_{k'}}{E_k}}_{\left(\frac{d\sigma}{d\Omega}\right)_{ns}} \times (1 - \tan^2\tfrac{\theta}{2}\frac{q^2}{2m_\mu^2})}$$

2- e-μ scattering

Remarks :

- The term $\left(\frac{d\sigma}{d\Omega}\right)_{ns}$ represents the scattering on a spin-less, structure-less particle (e.g. electron-pion w/o F.F.)
- There is an extra contribution $\propto \tan^2\frac{\theta}{2}\frac{q^2}{2m_\mu^2}$ due to the existence of the spin => coupling to the charge and the magnetic moment
- The above computation is transposable to the 'annihilation' case : $e^+ + e^- \to \mu^+ + \mu^-$

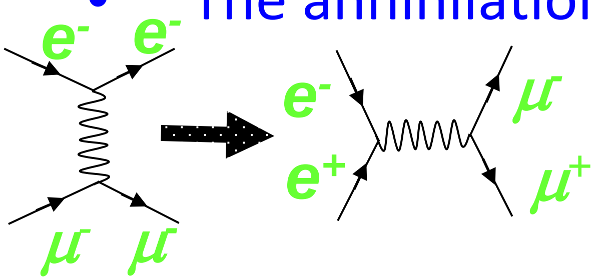
2- e-μ scattering

Remarks :

- In the CM frame (neglecting all leptons masses) one can express simply for the process $e^- + \mu^- \to e^- + \mu^-$:

$$|T|^2 = 2e^4 \frac{s^2+u^2}{t^2} \text{ with } \begin{cases} s = (k+p)^2 \\ t = (k-k')^2 \\ u = (k-p')^2 \end{cases}$$

- The annihilation process is obtained by substituting :



$$k' \to -p \Rightarrow s \to t \Rightarrow |T|^2 = 2e^4 \frac{t^2+u^2}{s^2}$$

$$\left(\frac{\partial\sigma}{\partial\Omega}\right)_{CM} = \frac{\alpha^2}{4s}\left(1+\cos^2\theta\right) \text{ and } \sigma = \frac{4\pi\alpha^2}{3s}$$

2- e-μ scattering

Remarks :

- Good agreement with experimental data (angular distribution and total cross-sections) <u>at lower energies</u>

- At higher energies one feels the effects of the weak interaction
  $\sqrt{s} \approx 90\ \mathrm{GeV}$