Chapter 6

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Electromagnetic interactions

Outline/Plan

1. Cross-section
2. Matrix element
3. Quantum ElectroDynamics (QED) Feynman rules

1. Section efficace
2. Elément de matrice
3. Règles de Feynman pour l'électro-dynamique quantique

1- Cross-section

Consider the general process for two-particle scattering :

a+b → 1+2+…+n

- States normalization : in NRQM usually one takes 1 particle per unit volume. For KG solutions :

$$\phi_a = N_a e^{-ip_a x} \Rightarrow \rho_a = i\left[\phi_a^*(\partial_t \phi_a) - (\partial_t \phi_a^*)\phi_a\right] = 2N_a^2 E_a$$

- Usual prescription (covariant normalization) :

2E particles per unit volume

$$\int_V d^3x \rho_a = 2E_a \Rightarrow N_a = V^{-1/2}$$

1- Cross-section

The general expression of the cross-section reads :

$$d\sigma = \frac{1}{2E_a 2E_b \left|\vec{v}_a - \vec{v}_b\right|} \int (2\pi)^4 \delta^4(p_f - p_i) \overline{\left|T_{fi}\right|^2} \prod_{k=1}^{k=n} \frac{d^3 p_k}{(2\pi)^3 2E_k}$$

- $2E_a 2E_b \left|\vec{v}_a - \vec{v}_b\right| / V^2$ is the flux factor (number of incident particles per unit area reaching the target per unit time × number of target particles) with normalization at *2E* particles per unit volume.

- $\prod_{k=1}^{k=n} \frac{V d^3 p_k}{(2\pi)^3 2E_k}$ is the final state available phase space (for one particle in a volume *V* with a momentum in d^3p one has the usual expression : $Vd^3 p / (2\pi)^3$

1- Cross-section

- $(2\pi)^4\delta^4(p_f - p_i)\overline{\left|T_{fi}\right|^2}$ is the transition rate per volume unit. It is the part containing the physics of the interaction processes.

  Each diagram's line is associated with a normalization factor N_i (i=a,b,1,…,n) which induces an overall factor :

$$\left[\underbrace{(V^{-1/2})^2}_{a,b} \times \underbrace{(V^{-1/2})^n}_{1\ldots n}\right]^2 = V^{-(n+2)}$$

  balancing the contributions of the flux and the phase space => the cross-section is independent of the normalization volume!

1- Cross-section

Application : Rutherford scattering formula (2+2 process => n=2).

- Flux factor : $2E_a 2E_b \left|\vec{v}_a - \vec{v}_b\right| = 4\left[(p_a.p_b)^2 - m_a^2 m_b^2\right] = 4\left|\vec{p}_a^{\,*}\right|\sqrt{s}$

  where the Mandelstam variable *s* has been used $s = \left(p_a + p_b\right)^2$

- 2-bodies phase space :

$$dQ_2 = \int_4 (2\pi)^4 \delta^4(p_f - p_i) \frac{d^3 p_1}{(2\pi)^3 2E_1} \frac{d^3 p_2}{(2\pi)^3 2E_2}$$

$$= \frac{1}{16\pi^2} \frac{\left|\vec{p}_1\right|^3 d\Omega_1}{\left|\vec{p}_1\right|^2 E_T - \vec{p}_1.\vec{p}_T E_1} \text{ with } (p_T) = (p_a + p_b)$$

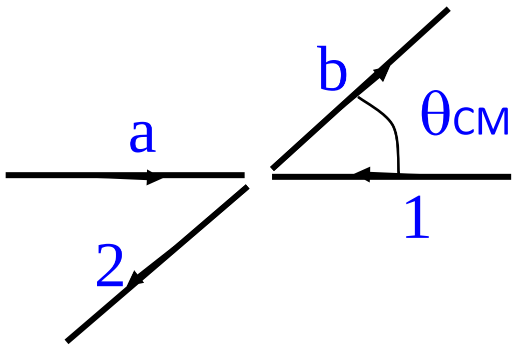
$$= \frac{1}{16\pi^2} \frac{\left|\vec{p}_1^{\,*}\right| d\Omega_1}{\sqrt{s}}$$

1- Cross-section

Application : Rutherford scattering formula (2+2 process => n=2).

- Transition rate :

$$\left|T_{fi}\right|^2 = \left(\frac{e^2}{q^2}\right)^2 \left(p_a + p_b\right).\left(p_1 + p_2\right)$$



- Mandelstam variables :

$$s = \left(p_a + p_1\right)^2 \quad t = \left(p_a - p_b\right)^2 \quad u = \left(p_a - p_2\right)^2$$

$$\left|T_{fi}\right|^2 = \left(\frac{4\pi\alpha}{t}\right)^2 \left(s-u\right)^2 \text{ and } \frac{d\sigma}{d\Omega} = \frac{1}{64\pi^2 s}\frac{\left|\vec{p}_1^{\,*}\right|}{\left|\vec{p}_a^{\,*}\right|}\left|T_{fi}\right|^2$$

- In the LAB frame (target at rest)

1- Cross-section

Application : Rutherford scattering formula (2+2 process => n=2).

- "Classical" case : *a* is a light particle (negligible *m*), *1* at rest initially => use of the LAB frame

$$p_a = \left(|\vec{k}|, \vec{k}\right) \quad p_1 = \left(M, \vec{0}\right)$$

$$t = -4|\vec{k}|^2 \sin^2(\theta/2) \quad (s-u)^2 = 16|\vec{k}|^2 M^2$$

$$4\left[(p_a.p_1)^2 - m^2 M^2\right] = 4|\vec{k}|^2 M^2$$

- With the fine structure constant $\alpha = e^2 / 4\pi$

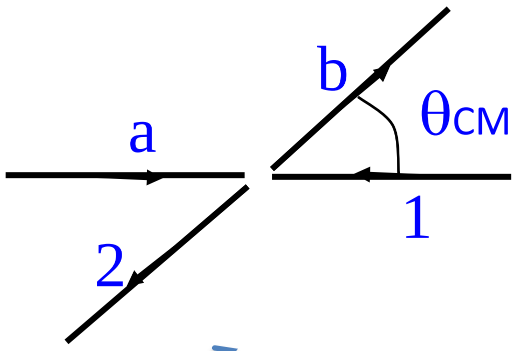
  one gets the standard Rutherford formula :

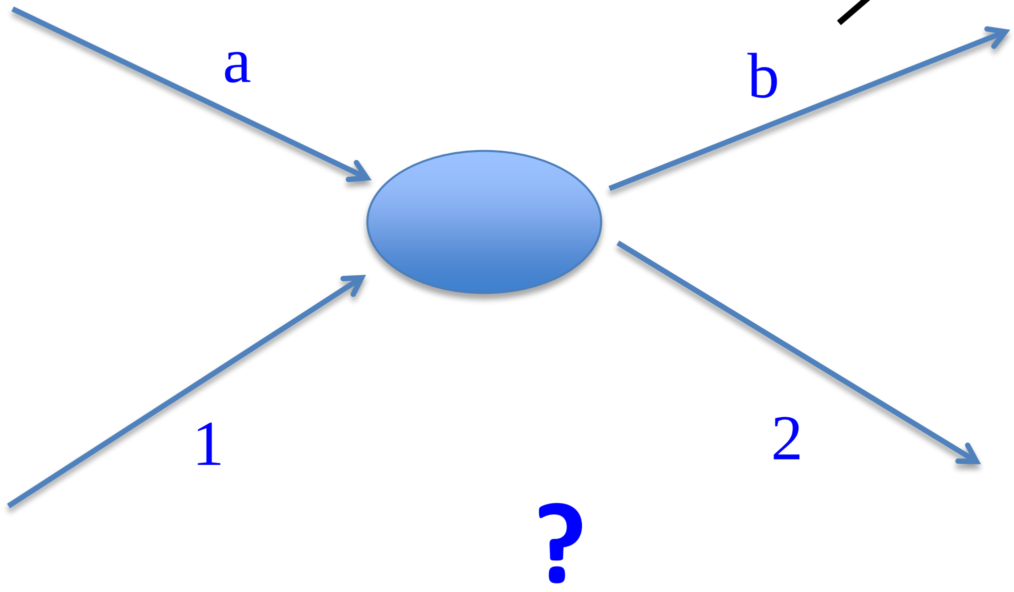
$$\frac{d\sigma}{d\Omega} = \frac{\alpha^2}{4|\vec{k}|^2 \sin^4(\theta/2)}$$

1- Cross-section

Question : how one computes the transition rate

$$\left|T_{fi}\right|^2 = \left(\frac{e^2}{q^2}\right)^2 \left(p_a + p_b\right).\left(p_1 + p_2\right)$$





2- Matrix element

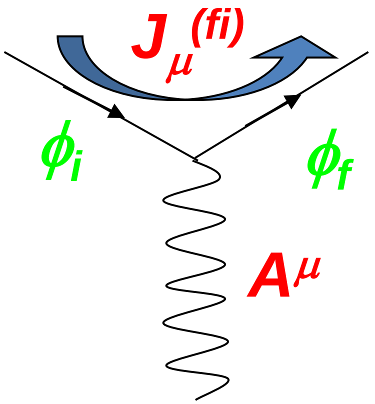
- Classical theory : transition amplitude (covariant expression) from initial *(i)* to final *(f)* state under the action of a perturbation potential *V.*

$$T_{fi} = -i\int d^4x\, \phi_f^*(x) V(x) \phi_i(x)$$

- 1st order K.G. potential : $V = -ie\left(\partial_\mu A^\mu + A^\mu \partial_\mu\right)$

- Integrating by parts introduces the transition current :

$$T_{fi} = -i\int d^4x\, A^\mu \underbrace{(-ie)\left[\phi_f^*\partial_\mu\phi_i - \partial_\mu\phi_f^*\phi_i\right]}_{J_\mu^{(fi)}}$$



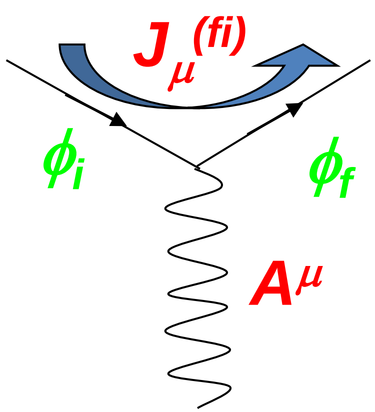
2- Matrix element

- *(i)* and *(f)* are taken as free particle solutions : $\phi_{i,f}(x) = N_{i,f} e^{-ip_{i,f}x}$

- Inserting the free particle solutions leads to

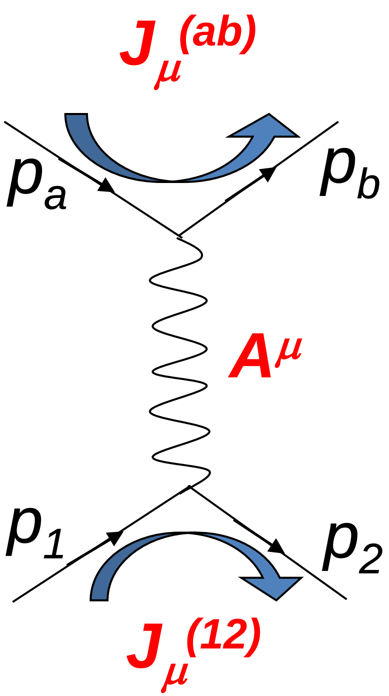
$$T_{fi} = -i\int d^4x\, A^{\mu}\left(-ie\right)\left[\phi_f^{*}\partial_{\mu}\phi_i - \partial_{\mu}\phi_f^{*}\phi_i\right]$$

$$= -i\int d^4x\, A^{\mu}(x)\, \underbrace{(-)eN_iN_f\left(p_i + p_f\right)_{\mu} e^{-i\left(p_i - p_f\right)x}}_{J_{\mu}^{(fi)}}$$



2- Matrix element

- Consider the process : $a+1 \rightarrow b+2$



The intermediate field is no more a free particle field but is induced by the motion of the bottom part of the diagram.

The A^μ field is given by the photon propagation equations in presence of a source :

$$\left(\partial^\nu \partial_\nu\right) A^\mu(x) - \partial^\mu \underbrace{\left(\partial^\nu A_\nu\right)}_{\substack{=0 \\ \text{in Lorentz} \\ \text{condition}}} = J^\mu_{(12)}$$

- The current is given by :

$$J_\mu^{(12)} = \langle p_2 | \hat{J}_\mu | p_1 \rangle = -eN_1 N_2 \left(p_1 + p_2\right)_\mu e^{-i(p_1 - p_2).x}$$

- The previous expression allows to compute the A^{μ} field :

$$\square A_{\mu} = J_{\mu}^{(12)} = -eN_1N_2\left(p_1+p_2\right)_{\mu} e^{-i(p_1-p_2).x} = -eN_1N_2\left(p_1+p_2\right)_{\mu} e^{iq.x}$$

$$\Rightarrow A_{\mu} = \langle 0|\hat{A}_{\mu}|q\rangle = +\frac{1}{q^2} eN_1N_2\left(p_1+p_2\right)_{\mu} e^{iq.x}$$

$$\Rightarrow A_{\mu} = \frac{-g_{\mu}^{\nu}}{q^2} J_{\nu}^{(12)}$$

- Replacing this expression in the matrix element :

$$\boxed{T_{fi} = -i\int d^4x\, J_{\mu}^{(ab)} \frac{-g^{\mu\nu}}{q^2} J_{\nu}^{(12)}}$$

- Namely :

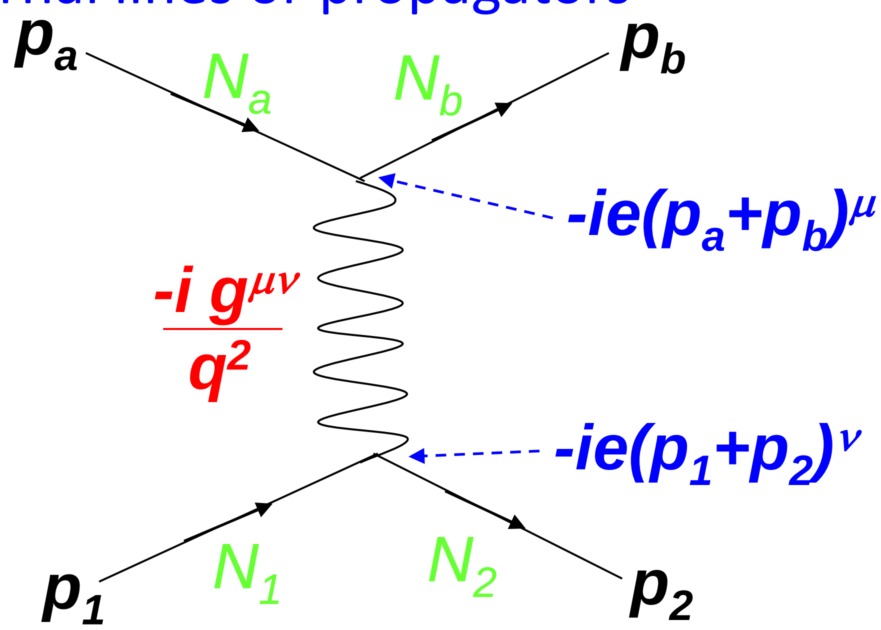
$$T_{fi} = -i(-e)(-e)N_aN_bN_1N_2\left(p_a+p_b\right)_{\mu} \frac{-g^{\mu\nu}}{q^2}\left(p_1+p_2\right)_{\nu}$$

$$\times(2\pi)^4\delta^4(p_f - p_i - k)$$

3- Feynman rules for QED

Summary : "reading" a Feynman diagram

- The basic elements of a diagram are :
  - the external lines
  - the vertex operator
  - the internal lines or propagators



with the transferred momentum : $q = \left(p_a - p_b\right) = \left(p_2 - p_1\right)$

3- Feynman rules for QED

Including fermions into the game: same procedure

$\Rightarrow$ matrix element reduced expression

$$T_{fi} = -i\int d^4x\, J_\mu^{(ab)} \frac{-g^{\mu\nu}}{q^2} J_\nu^{(12)}$$

$\Rightarrow$ with a current given by (see lecture on Dirac's equation) :

$$J_\mu^{(fi)} = -e\left(\overline{\psi}_f \gamma_\mu \psi_i\right) = -e\left(\overline{u}_f \gamma_\mu u_i\right) e^{i(p_f - p_i).x}$$

$\Rightarrow$ N.B. to be compared with the spinless case :

$$J_\mu^{(fi)} = -e\left(p_f + p_i\right)_\mu e^{i(p_f - p_i).x}$$

3- Feynman rules for QED

Feynman rules for QED:

- External lines:

  - fermions =>

    (incoming)

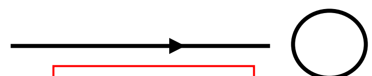
    $u(p,s)$

    (outgoing)

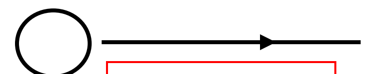
    

    $\overline{u}(p',s')$

  - anti-fermions =>

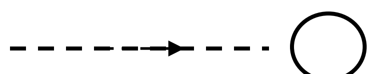
    

    $\overline{v}(p,s)$

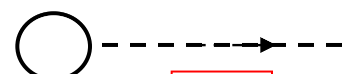
    

    $v(p',s')$

  - spin-0 bosons =>

    cst

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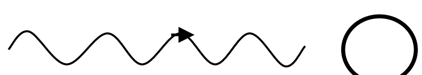
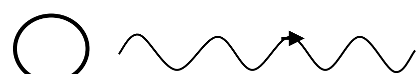
  - photons =>

    $\varepsilon_\mu(k,\lambda)$

    $\varepsilon^*_\mu(k',\lambda')$

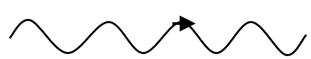
3- Feynman rules for QED

Feynman rules for QED:

- Propagators:

  - fermions =>  $$\frac{i}{\not p - m} = \frac{i(\not p + m)}{p^2 - m^2}$$

  - spin-0 bosons =>  $$\frac{i}{p^2 - m^2}$$

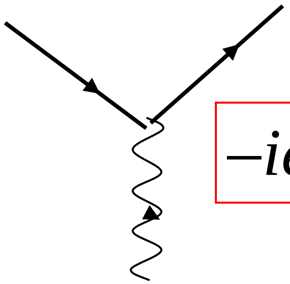
  - photons =>  $$\frac{-ig^{\mu\nu}}{p^2}$$

3- Feynman rules for QED

Feynman rules for QED:

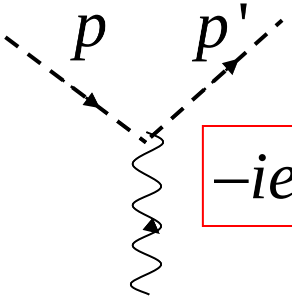
- Vertices :

  - fermions – photon =>



$$-ie\gamma^{\mu}$$

  - spin-0 bosons – photon =>



$$-ie(p+p')^{\mu}$$