


***Chapter 5***  
  
***Dirac equation***

# Outline/Plan

## Introduction

- The non relativistic case
- The Dirac equation
- Adjoint representation
- Quadri-current
- Dirac spinors
- Feynman's prescription

## Summary

## Introduction

- Le cas non-relativiste
- L'équation de Dirac
- Représentation adjointe
- Quadri-courant
- Spineurs de Dirac
- Prescription de Feynman

## Résumé

# Introduction

The non-relativistic case : the Pauli equation

- Preliminary remark on Pauli matrices :

$$\sigma_1 = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} \quad \sigma_2 = \begin{pmatrix} 0 & -i \\ i & 0 \end{pmatrix} \quad \sigma_3 = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}$$

if  $\vec{O}_1$  and  $\vec{O}_2$  are operators commuting with the Pauli matrices then

$$(\vec{\sigma} \cdot \vec{O}_1)(\vec{\sigma} \cdot \vec{O}_2) = \vec{O}_1 \cdot \vec{O}_2 + i\vec{\sigma} \cdot (\vec{O}_1 \times \vec{O}_2)$$

- Applying the previous relation to  $\vec{P}$  leads to :

$$(\vec{\sigma} \cdot \vec{P})(\vec{\sigma} \cdot \vec{P}) = (\vec{\sigma} \cdot \vec{P})^2 = \vec{P}^2 \Rightarrow H = \frac{(\vec{\sigma} \cdot \vec{P})^2}{2m}$$

# Introduction

- Reminder (Pauli matrices properties) :
  - $\{\sigma_i, \sigma_j\} = 2\delta_{ij} \quad [\sigma_i, \sigma_j] = 2i\epsilon_{ijk}\sigma_k$
  - $\sigma_i\sigma_j = \delta_{ij} + i\epsilon_{ijk}\sigma_k$
  - $(\vec{\sigma}\cdot\vec{O}_1)(\vec{\sigma}\cdot\vec{O}_2) = \vec{O}_1\cdot\vec{O}_2 + i\vec{\sigma}\cdot(\vec{O}_1\times\vec{O}_2)$
  - For  $\vec{u}$  unit vector  $\frac{1}{2}(1\pm\vec{\sigma}\cdot\vec{u})\chi$  eigenfunction of  $\vec{\sigma}\cdot\vec{u}$  with eigenvalue  $\pm 1 \quad \forall\chi$ .
  - $\exp\left(-i\frac{\theta}{2}\sigma_k\right) = \cos\left(\frac{\theta}{2}\right) - i\sigma_k \sin\left(\frac{\theta}{2}\right)$

# 1- The non-relativistic case

- To describe the particle's motion in an E.M. field the covariant derivative prescription leads to :

$$p^\mu \rightarrow p^\mu + eA^\mu \text{ i.e. } \begin{cases} i\hbar\partial_t \rightarrow i\hbar\partial_t + eA^0 \\ \vec{p} \rightarrow \vec{p} + e\vec{A} \end{cases}$$

- The Schrödinger equation

$$i\hbar\frac{\partial\psi}{\partial t}(\vec{x},t) = \frac{\vec{P}^2}{2m}\psi(\vec{x},t)$$

can be rewritten as

$$i\hbar\frac{\partial\psi}{\partial t}(\vec{x},t) + eA^0\psi(\vec{x},t) = \frac{\left(\vec{\sigma}\cdot(\vec{P} + e\vec{A})\right)^2}{2m}\psi(\vec{x},t)$$

# 1- The non-relativistic case

- Using the Pauli matrices relation :

$$\left(\vec{\sigma} \cdot (\vec{P} + e\vec{A})\right)^2 \psi = (\vec{P} + e\vec{A})^2 \psi + i\vec{\sigma} \cdot \left( (\vec{P} + e\vec{A}) \times (\vec{P} + e\vec{A}) \right) \psi$$

- The last term reads :

$$\begin{aligned} \left( (\vec{P} + e\vec{A}) \times (\vec{P} + e\vec{A}) \right)_i \psi &= e \left( \vec{P} \times \vec{A} + \vec{A} \times \vec{P} \right)_i \psi \\ &= e \varepsilon_{ijk} \left( \vec{P}_j \vec{A}_k + \vec{A}_j \vec{P}_k \right) \psi \\ &\Rightarrow -e i\hbar \varepsilon_{ijk} \left( \vec{\partial}_j \vec{A}_k + \vec{A}_j \vec{\partial}_k \right) \psi \end{aligned}$$

# 1-The non-relativistic case

- Developing the last term :

$$-e i\hbar \varepsilon_{ijk} \left( \vec{\partial}_j \vec{A}_k + \vec{A}_j \vec{\partial}_k \right) \psi =$$

$$-e i\hbar \varepsilon_{ijk} \left( \vec{\partial}_j \left( \vec{A}_k \psi \right) + \vec{A}_j \left( \vec{\partial}_k \psi \right) \right) =$$

$$-e i\hbar \varepsilon_{ijk} \left( \left( \vec{\partial}_j \vec{A}_k \right) \psi + \underbrace{\vec{A}_k \left( \vec{\partial}_j \psi \right) + \vec{A}_j \left( \vec{\partial}_k \psi \right)}_{=0} \right) =$$

$$-e i\hbar \varepsilon_{ijk} \left( \vec{\partial}_j \vec{A}_k \right) \psi =$$

$$-e i\hbar \left( \vec{\nabla} \times \vec{A} \right)_i \psi =$$

$$e i\hbar \vec{B}_i \psi$$

# 1- The non-relativistic case

- Finally the Pauli equation reads

$$i\hbar \frac{\partial \psi}{\partial t}(\vec{x}, t) = \frac{(\vec{P} + e\vec{A})^2}{2m} \psi(\vec{x}, t) - \frac{e\hbar}{2m} (\vec{\sigma} \cdot \vec{B}) \psi(\vec{x}, t) - eA^0 \psi(\vec{x}, t)$$

where the wavefunction is a 2D object.

- The action of the **spin operator** is explicit.
- N.B. developing  $\vec{P} + e\vec{A}$  leads to  $\vec{L} \cdot \vec{B}$ . One gets the complete equation through the substitution  $\vec{L} \rightarrow \vec{L} + 2\vec{S}$  with  $\vec{S} = \frac{\hbar}{2} \vec{\sigma}$



## 2- The Dirac equation

- Generalization to the relativistic case : the starting point is always

$$E^2 = \vec{p}^2 c^2 + m^2 c^4$$

which is the only covariant form for the canonical quantization :

$$\left[ (i\hbar\partial_t)^2 - (\vec{\sigma}\cdot\vec{P})^2 \right] \psi(\vec{x}, t) = m^2 \psi(\vec{x}, t)$$

$$\left[ i\hbar\partial_t + i\hbar\vec{\sigma}\cdot\vec{\nabla} \right] \left[ i\hbar\partial_t - i\hbar\vec{\sigma}\cdot\vec{\nabla} \right] \psi = m^2 \psi$$

- Let's take auxiliary variables :

$$\psi^{(1)} = \psi$$

$$\psi^{(2)} = \frac{1}{m} \left[ i\hbar\partial_t - i\hbar\vec{\sigma}\cdot\vec{\nabla} \right] \psi$$

## 2- The Dirac equation

- The following system

$$\begin{cases} \left[ i\hbar\partial_t - i\hbar\vec{\sigma}\cdot\vec{\nabla} \right] \psi^{(1)} = m\psi^{(2)} \\ \left[ i\hbar\partial_t + i\hbar\vec{\sigma}\cdot\vec{\nabla} \right] \psi^{(2)} = m\psi^{(1)} \end{cases}$$

is equivalent to

$$\begin{cases} i\hbar\partial_t\varphi + i\hbar\vec{\sigma}\cdot\vec{\nabla}\chi = m\varphi \\ -i\hbar\partial_t\chi - i\hbar\vec{\sigma}\cdot\vec{\nabla}\varphi = m\chi \end{cases} \quad \text{with} \quad \begin{cases} \varphi = \psi^{(2)} + \psi^{(1)} \\ \chi = \psi^{(2)} - \psi^{(1)} \end{cases}$$

# 2- The Dirac equation

- In a matrix form this equation reads :

$$\begin{pmatrix} i\hbar\partial_t & i\hbar\vec{\sigma}\cdot\vec{\nabla} \\ -i\hbar\vec{\sigma}\cdot\vec{\nabla} & -i\hbar\partial_t \end{pmatrix} \begin{pmatrix} \varphi \\ \chi \end{pmatrix} = m \begin{pmatrix} \varphi \\ \chi \end{pmatrix}$$
$$\Leftrightarrow \begin{pmatrix} i\hbar\partial_t & i\hbar\vec{\sigma}\cdot\vec{\nabla} \\ -i\hbar\vec{\sigma}\cdot\vec{\nabla} & -i\hbar\partial_t \end{pmatrix} \psi_{4D} = m\psi_{4D}$$

with

$$\psi_{4D} \equiv \begin{pmatrix} \varphi \\ \chi \end{pmatrix} = \begin{pmatrix} \psi_1 \\ \psi_2 \\ \psi_3 \\ \psi_4 \end{pmatrix}$$

## 2- The Dirac equation

- Gamma matrices definition :

$$\gamma^0 = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix} = \gamma_0 \quad \text{and} \quad \vec{\gamma}^i = \begin{pmatrix} 0 & \sigma^i \\ -\sigma^i & 0 \end{pmatrix} = -\vec{\gamma}_i$$

- Basic properties :

$$\{\gamma^\mu, \gamma^\nu\} = \gamma^\mu \gamma^\nu + \gamma^\nu \gamma^\mu = 2g^{\mu\nu}$$

$$(\gamma^0)^2 = 1 \quad \text{and} \quad (\gamma^i)^2 = -1$$

$$\gamma^{\mu\dagger} = \gamma^0 \gamma^\mu \gamma^0$$

- “Slash” notation :  $\not{a} = a_\mu \gamma^\mu$

## 2- The Dirac equation

- The previous matrix equation writes :

$$\begin{pmatrix} i\hbar\partial_t & i\hbar\vec{\sigma}\cdot\vec{\nabla} \\ -i\hbar\vec{\sigma}\cdot\vec{\nabla} & -i\hbar\partial_t \end{pmatrix} \begin{pmatrix} \varphi \\ \chi \end{pmatrix} = m \begin{pmatrix} \varphi \\ \chi \end{pmatrix} \Leftrightarrow i\hbar(\gamma^0\partial_0 + \gamma^i\partial_i)\psi = m\psi$$

$$(i\hbar\gamma^\mu\partial_\mu - m)\psi = 0 \quad \text{Dirac Equation}$$

- In details :

$$\sum_{\mu=0,1,2,3} \sum_{b=1,2,3,4} (i\hbar(\gamma^\mu)_{ab} \partial_\mu - m\delta_{ab}) \psi_b = 0$$

# 3- Adjoint representation

- Let's start from the "conjugate" equation :

$$(i\hbar\gamma^\mu\partial_\mu - m)\psi = 0 \Rightarrow \psi^\dagger (i\hbar\gamma^{\mu\dagger}\bar{\partial}_\mu + m) = 0$$

- Using the gamma matrices property :  $\gamma^{\mu\dagger} = \gamma^0\gamma^\mu\gamma^0$  one gets

$$\psi^\dagger (i\hbar\gamma^0\gamma^\mu\gamma^0\bar{\partial}_\mu + m\gamma^0\gamma^0) = 0$$

$$\psi^\dagger\gamma^0 (i\hbar\gamma^\mu\gamma^0\bar{\partial}_\mu + m\gamma^0) = 0 \quad \leftarrow (\times\gamma^0)$$

$$\bar{\psi} (i\hbar\gamma^\mu\bar{\partial}_\mu + m) = 0 \quad \text{where } \boxed{\bar{\psi} = \psi^\dagger\gamma^0}$$

# 3- Adjoint representation

- One defines the **adjoint spinor**  $\bar{\psi} = \psi^\dagger \gamma^0$

- The conjugate equations read therefore :

$$(i\hbar\gamma^\mu\partial_\mu - m)\psi = 0$$

and

$$\bar{\psi}(i\hbar\gamma^\mu\bar{\partial}_\mu + m) = 0$$

- Reminder : spinless case (K.G.)

$$\partial^\mu\partial_\mu\psi + m^2\psi = 0$$

and

$$\partial^\mu\partial_\mu\psi^* + m^2\psi^* = 0$$

# 4. Quadri-current

- Formal derivation possibilities :
  - directly from the fundamental and adjoint representations or
  - from the Lagrangian expression  $\oplus$  gauge invariance (looking for the current coupling to the gauge field)
- Question : is the Born probabilistic interpretation possible in the Dirac case (i.e. is the charge density positive to be interpreted as a probability density?).
- Reminder : not the case in the spinless case where negative energy solutions and negative charge density should have been re-interpreted.

$$\psi(x) = Ne^{i(\vec{p}\cdot\vec{x}-Et)/\hbar} \Rightarrow \rho = 2|N|^2 \times \underbrace{E}_{>0 \text{ or } 0<}$$



# 4. Quadri-current

Direct derivation :

$$\left. \begin{aligned} (i\hbar\gamma^\mu\partial_\mu - m)\psi &= 0 \\ \bar{\psi}(i\hbar\gamma^\mu\overleftarrow{\partial}_\mu + m) &= 0 \end{aligned} \right\} \oplus$$

$$\bar{\psi}\gamma^\mu(\overleftarrow{\partial}_\mu + \partial_\mu)\psi = 0 \Leftrightarrow \partial_\mu(\bar{\psi}\gamma^\mu\psi) = 0$$

- Conserved current :  $J^\mu \equiv c(\bar{\psi}\gamma^\mu\psi)$
- Conserved charge density :  $\rho \equiv J^0/c = \bar{\psi}\gamma^0\psi = \psi^\dagger \underbrace{\gamma^0\gamma^0}_{=1}\psi$   
 $\Rightarrow \rho = \psi^\dagger\psi = \sum_{a=1,2,3,4} \psi_a^*\psi_a > 0$

# 4. Quadri-current

## Derivation from gauge invariance :

- Lagrangian of free particles :  $L_{free} = \bar{\psi} (i\gamma^\mu \partial_\mu - m) \psi$
- $U(1)$  invariance  $\rightarrow$  use of covariant derivatives :

$$\begin{aligned} L_{int.} &= \bar{\psi} (i\gamma^\mu D_\mu - m) \psi \\ &= \bar{\psi} (i\gamma^\mu (\partial_\mu + ieA_\mu) - m) \psi \\ &= \bar{\psi} (i\gamma^\mu \partial_\mu - m) \psi - \underbrace{e\bar{\psi}\gamma^\mu\psi}_{\propto J^\mu} A_\mu \end{aligned}$$

where the interaction term appears on the form  $-J_\mu A^\mu$  as for spinless particles.

# 5. Dirac spinors

## Free particle solutions.

- By analogy with the non-relativistic case we may write a general spin-½ wavefunction in terms of factorized solutions :

$$\begin{aligned}\psi &= u \times (\text{plane wave}) \\ &= u \times \exp(-ip^\mu x_\mu) = u \times \exp(-ipx) \\ &= u(p)e^{[i(\vec{p}\cdot\vec{x}-Et)]}\end{aligned}$$

where  $u$  is a 4-components **spinor**.

- The explicit expression of the spinor is more easily deduced from the Hamiltonian formalism of the Dirac equation...

# 5. Dirac spinors

- The solution to the Dirac equation may be written as :

$$u = N_p \begin{pmatrix} u_A \\ \frac{\vec{\sigma} \cdot \vec{p}}{E + m} u_A \end{pmatrix}$$

where  $u_A$  is a 2D spinors  $u_A = \begin{pmatrix} 1 \\ 0 \end{pmatrix}$  or  $u_A = \begin{pmatrix} 0 \\ 1 \end{pmatrix}$

- The  $E - p$  relation reads :

$$(\vec{\sigma} \cdot \vec{p})(\vec{\sigma} \cdot \vec{p})u_A = \vec{p}^2 u_A = (E - m)(E + m)u_A \Rightarrow E = \pm E_p$$

# 5. Dirac spinors

## Summary.

- Positive-energy solutions :

$$\psi^{(+)(s=1,2)}(x) \equiv u^{(s=1,2)}(p) e^{-ipx} \quad u^{(s)}(p) \propto \begin{pmatrix} \varphi^{(s)} \\ \frac{\vec{\sigma} \cdot \vec{p}}{E + m} \varphi^{(s)} \end{pmatrix}$$

- Negative-energy solutions :

$$\psi^{(-)(s=1,2)}(x) \equiv v^{(s=1,2)}(p) e^{+ipx} \quad v^{(s)}(p) \propto \begin{pmatrix} \frac{\vec{\sigma} \cdot \vec{p}}{E + m} \chi^{(s)} \\ \chi^{(s)} \end{pmatrix}$$

with the spin-up/down 2D spinors  $\varphi, \chi^{(s=1)} = \begin{pmatrix} 1 \\ 0 \end{pmatrix}$  and  $\varphi, \chi^{(s=2)} = \begin{pmatrix} 0 \\ 1 \end{pmatrix}$

- Warning on spin direction : negative-energy solution with spin- $\uparrow$  = positive-energy solution with spin- $\downarrow$

# 5. Dirac spinors

- Dirac equation in  $p$ -space :

$$\left(\gamma^\mu p_\mu - m\right)u(p) = 0$$

$$\left(\gamma^\mu p_\mu + m\right)v(p) = 0$$

- Adjoint spinors (same procedure as in  $x$ -space) :

$$\bar{u}(p)\left(\gamma^\mu p_\mu - m\right) = 0$$

$$\bar{v}(p)\left(\gamma^\mu p_\mu + m\right) = 0$$

- Ortho-normalization relations :

$$\bar{u}^{(s)}(p)u^{(s')}(p) = \delta_{ss'}, \quad \bar{v}^{(s)}(p)v^{(s')}(p) = -\delta_{ss'}$$

$$\bar{u}^{(s)}(p)v^{(s')}(p) = 0 \quad \bar{v}^{(s)}(p)u^{(s')}(p) = 0$$

# 5. Dirac spinors

Some useful relations :

$$\sum_{s=1,2} u^{(s)}(p) \bar{u}^{(s)}(p) = (\not{p} + m)$$
$$\sum_{s=1,2} v^{(s)}(p) \bar{v}^{(s)}(p) = (\not{p} - m)$$

- 'Energy' projectors :

$$\Lambda_+ = \frac{(\not{p} + m)}{2m} \text{ such as } \Lambda_+ u = u \text{ and } \Lambda_+ v = 0$$

$$\Lambda_- = \frac{(-\not{p} + m)}{2m} \text{ such as } \Lambda_- v = v \text{ and } \Lambda_- u = 0$$

$$\Lambda_{\pm}^2 = \Lambda_{\pm} \text{ and } \Lambda_+ + \Lambda_- = 1$$

# 6. Feynman's prescription

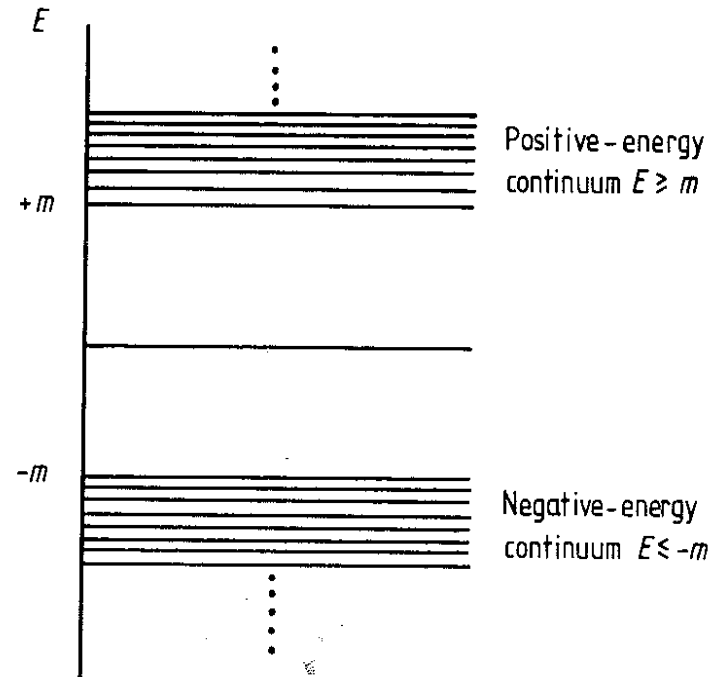
## Summary.

- Klein-Gordon equation  $\rightarrow$   $\left\{ \begin{array}{l} \text{negative probabilities} \\ \text{negative energies} \end{array} \right.$
- Dirac equation  $\rightarrow$   $\left\{ \begin{array}{l} \text{positive (only) probabilities} \\ \text{negative energies} \end{array} \right.$
- Dirac's historical interpretation : the 'vacuum' state consists of all negative-energy states filled with electrons. The Pauli principle forbids any positive-energy electron from falling into these lower energy states.



# 6. Feynman's prescription

- The 'vacuum' (so-called Dirac sea) has now infinite negative charge and energy but all observations represent finite fluctuations w.r.t. the vacuum.



- A 'hole' in the Dirac sea, i.e. the absence of a negative-energy electron is equivalent to the presence of a positive-energy positively charged version of the electron, namely a positron.

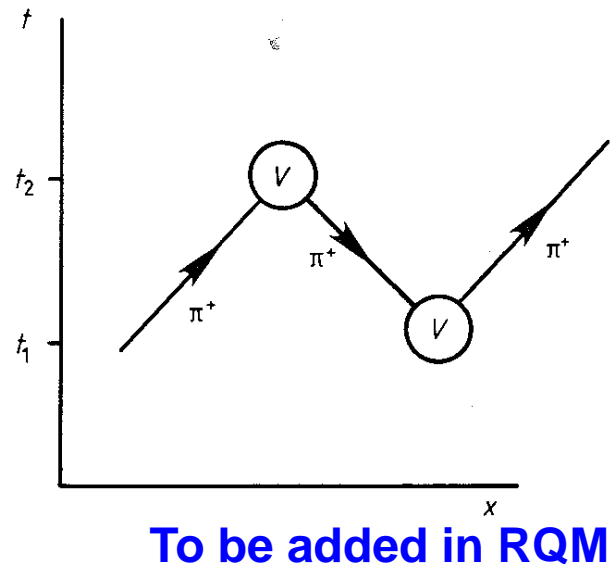
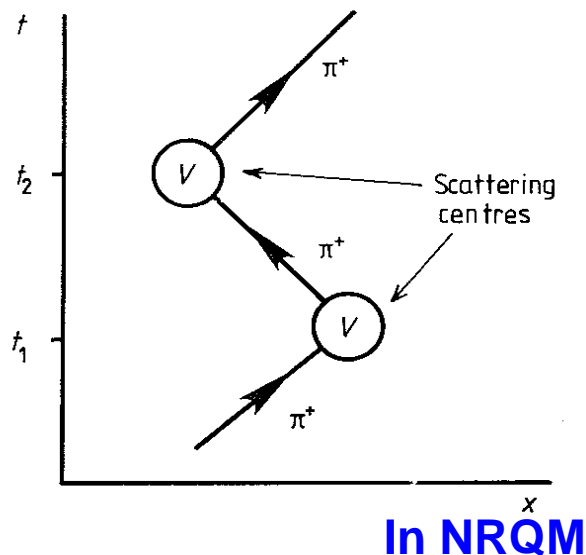
$$\text{energy of 'hole'} = -\left(E_{neg}\right) \rightarrow \text{positive energy}$$

$$\text{charge of 'hole'} = -\left(q_e\right) \rightarrow \text{positive charge}$$

# 6. Feynman's prescription

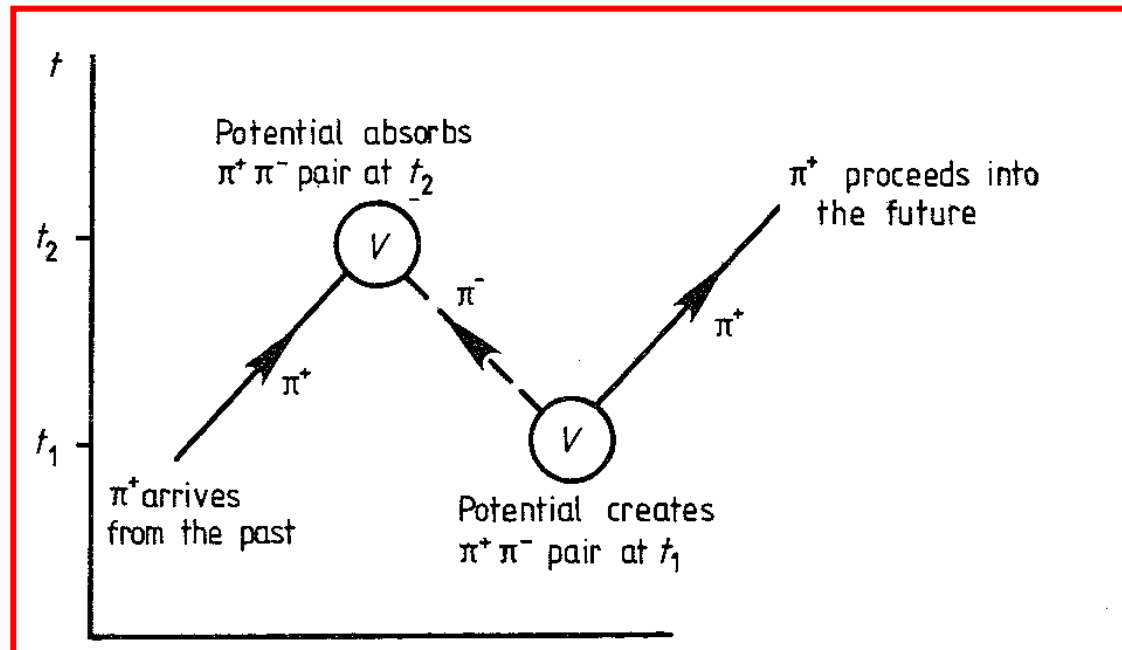
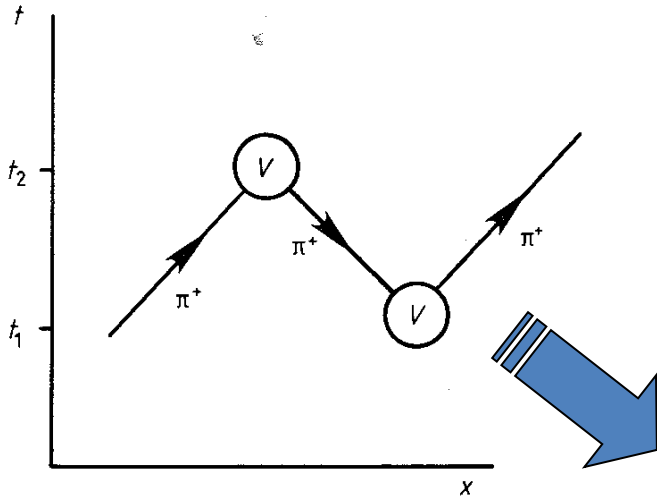
- What about solutions to the Klein-Gordon equation (bosons not affected by Pauli principle)?
- Interpretation through Feynman's prescription (1962) :

negative-energy particle solutions propagating backward in time **L**  
positive-energy antiparticle solutions propagating forward in time



# 6. Feynman's prescription

- Consequence of the Feynman prescription :



# Summary

- Relativistic propagation equations derived from energy definition law  $\Rightarrow$  Klein-Gordon (bosons) and Dirac (fermions)
- Gauge invariance imposed to the theory  $\Rightarrow$  existence of conserved currents coupled to gauge fields (e.g. photons for EM)
- From classical to quantum field theory : fields  $\Rightarrow$  operators
- Lagrangian formalism widely used.