Chapter 4

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Electromagnetic interactions

Outline/Plan

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   1. Matrix element expression
   2. EM scattering of spinless bosons
   3. Cross-section

2. Feynman rules for QED

3. Applications to some processes
   1. Basic process
   2. Trace techniques for spin summations
   3. Expression of the cross-section
   4. 2^{nd} process : electron-muon scattering
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   2. Diffusion EM de bosons sans spin
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3. Applications à qqs processus
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   4. $2^{ème}$ processus : diffusion électron-muon
   5. Diffusion electron-nucleon

1-1 Matrix element

- Classical theory : transition amplitude (covariant expression) from initial *(i)* to final *(f)* state under the action of a perturbation potential *V*.

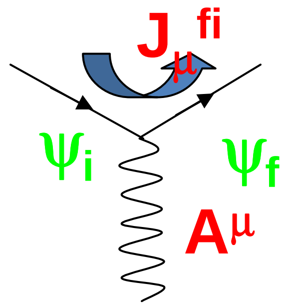
$$T_{fi} = -i\int d^4x\, \phi_f^*(x)V(x)\phi_i(x)$$

- 1st order K.G. potential : $V = -ie\left(\partial_\mu A^\mu + A^\mu\partial_\mu\right)$

- *(i)* and *(f)* are taken as free particle solutions : $\psi_{i,f}(x) = N_{i,f}e^{-ip_{i,f}x}$

- Integrating by parts introduces the transition current :

$$T_{fi} = -i\int d^4x\, A^\mu \underbrace{(-ie)\left[\phi_f^*\partial_\mu\phi_i - \partial_\mu\phi_f^*\phi_i\right]}_{J_\mu^{fi}}$$



1-1 Matrix element

- Inserting the free particle solutions leads to

$$T_{fi} = -i\int d^4x\, A^\mu \left(-ie\right)\left[\phi_f^* \partial_\mu \phi_i - \partial_\mu \phi_f^* \phi_i\right]$$

$$= +iN_i N_f \int d^4x\, A^\mu(x)\, e\left(p_i + p_f\right)_\mu e^{-i\left(p_i - p_f\right)x}$$

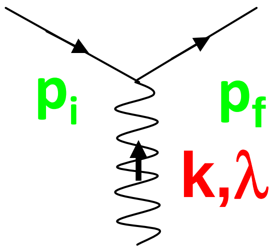
- What form for the A^μ field? For an incoming photon with momentum k and polarization state λ, we take :

$$A^\mu(x) = N_k \varepsilon^\mu(\lambda) e^{-ikx}$$

where ε^μ represents the polarization wavefunction

- Finally :

$$T_{fi} = +ieN_i N_f N_k \left(p_i + p_f\right).\varepsilon \underbrace{\int d^4x\, e^{-i\left(p_i - p_f\right)x} e^{-ikx}}_{(2\pi)^4 \delta^4(p_f - p_i - k)}$$



1-1 Matrix element

- What is the link with the quantum field theory? The analogue of the previous transition amplitude is :

$$T_{fi} = -i\int d^4x \left\langle f \left| \hat{H}^I_{\text{int}}(\vec{x},t) \right| i \right\rangle$$

  where the above operator is taken in 'interaction picture'.

- In present case the Hamiltonian may be expressed as :

$$\hat{H}^I_{\text{int}} = \hat{A}^\mu \hat{J}_\mu \text{ with } \hat{J}_\mu = -ie\left[\hat{\phi}^\dagger \partial_\mu \hat{\phi} - \left(\partial_\mu \hat{\phi}^\dagger\right)\hat{\phi}\right]$$

  where the operator field Φ expands over the elementary creation/annihilation operators :

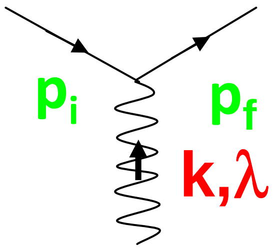
$$\hat{\phi}(x) = \int \frac{d^3p}{(2\pi)^3 2E}\left[\hat{a}(p)e^{-ipx} + \hat{b}^\dagger(p)e^{ipx}\right] \text{ with } \hat{a}^\dagger(p)\left|0\right\rangle = \left|p\right\rangle$$

1-1 Matrix element

- Short (very short…) cut : the QFT analog expression of the transition amplitude reads

$$T_{fi} = -i\int d^4x \left\langle p_f \middle| \hat{J}_\mu \middle| p_i \right\rangle \left\langle 0 \middle| \hat{A}^\mu \middle| k,\lambda \right\rangle$$

- The rules attached to the 1-photon absorption (emission) diagram can be deduced from the previous transition amplitude expression :

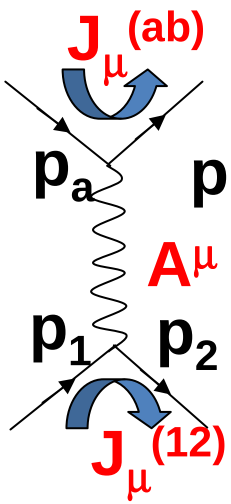


$$T_{fi} = +ieN_i N_f N_k \left(p_i + p_f\right).\varepsilon(2\pi)^4 \delta^4(p_f - p_i - k)$$

- $\exists$ a normalisation factor for each line of the diagram
- there is a vertex factor $ie\left(p_i + p_f\right)^\mu$ for charge *e*
- polarization vector is ε^μ ($\varepsilon^{\mu *}$) for incoming (outgoing) photon
- those factors are multiplied by a 4-momentum conservation δ function

1-2 EM scattering of spinless bosons

- Consider the process : $a + 1 \rightarrow b + 2$



The intermediate field is no more a free particle field but is induced by the motion of the bottom part of the diagram.

The A^μ field is given by the photon propagation equations in presence of a source :

$$\left(\partial^\nu \partial_\nu\right) A^\mu(x) - \partial^\mu \underbrace{\left(\partial^\nu A_\nu\right)}_{\substack{=0 \\ \text{in Lorentz} \\ \text{condition}}} = J^\mu_{(12)}$$

- The current is given by :

$$J_\mu^{(12)} = \left\langle p_2 \right| \hat{J}_\mu \left| p_1 \right\rangle = -eN_1 N_2 \left(p_1 + p_2\right)_\mu e^{-i(p_1 - p_2).x}$$

1-2 EM scattering of spinless bosons

- The previous expression allows to compute the A^μ field :

$$\Box A_\mu = J_\mu^{(12)} = -eN_1N_2\left(p_1+p_2\right)_\mu e^{-i(p_1-p_2).x} = -eN_1N_2\left(p_1+p_2\right)_\mu e^{iq.x}$$

$$\Rightarrow A_\mu = \langle 0|\hat{A}_\mu|q\rangle = +\frac{1}{q^2}eN_1N_2\left(p_1+p_2\right)_\mu e^{iq.x}$$

$$\Rightarrow A_\mu = \frac{-g_\mu^\nu}{q^2}J_\nu^{(12)}$$

- Replacing this expression in the matrix element :

$$T_{fi} = -i\int d^4x\, J_\mu^{(ab)}\frac{-g^{\mu\nu}}{q^2}J_\nu^{(12)}$$

- Namely :

$$T_{fi} = -i(-e)(-e)N_aN_bN_1N_2\left(p_a+p_b\right)_\mu\frac{-g^{\mu\nu}}{q^2}\left(p_1+p_2\right)_\nu$$

$$\times(2\pi)^4\delta^4(p_f-p_i-k)$$

1-3 Cross-section

Consider the general process for two-particle scattering :

a+b → 1+2+…+n

- States normalization : in NRQM usually one takes 1 particle per unit volume. For KG solutions :

$$\phi_a = N_a e^{-ip_a x} \Rightarrow \rho_a = i\left[\phi_a^*(\partial_t\phi_a) - (\partial_t\phi_a^*)\phi_a\right] = 2N_a^2 E_a$$

- Usual prescription (covariant normalization) :

2E particles per unit volume

$$\int_V d^3x \rho_a = 2E_a \Rightarrow N_a = V^{-1/2}$$

1-3 Cross-section

The general expression of the cross-section reads :

$$d\sigma = \frac{1}{2E_a 2E_b |\vec{v}_a - \vec{v}_b|} \int (2\pi)^4 \delta^4(p_f - p_i) \overline{|T_{fi}|^2} \prod_{k=1}^{k=n} \frac{d^3 p_k}{(2\pi)^3 2E_k}$$

- $2E_a 2E_b |\vec{v}_a - \vec{v}_b| / V^2$ is the flux factor (number of incident particles per unit area reaching the target per unit time $\times$ number of target particles) with normalization at *2E* particles per unit volume.

- $\prod_{k=1}^{k=n} \frac{V d^3 p_k}{(2\pi)^3 2E_k}$ is the final state available phase space (for one particle in a volume *V* with a momentum in d^3p one has the usual expression : $V d^3 p / (2\pi)^3$

1-3 Cross-section

- $(2\pi)^4\delta^4(p_f - p_i)\overline{|T_{fi}|^2}$ is the transition rate per volume unit. It is the part containing the physics of the interaction processes.

  Each diagram's line is associated with a normalization factor N_i (i=a,b,1,…,n) which induces an overall factor :

$$\left[\underbrace{(V^{-1/2})^2}_{a,b}\times\underbrace{(V^{-1/2})^n}_{1\ldots n}\right]^2 = V^{-(n+2)}$$

  balancing the contributions of the flux and the phase space => the cross-section is independent of the normalization volume!

1-3 Cross-section

Application : Rutherford scattering formula (2+2 process => n=2).

- Flux factor : $2E_a 2E_b \left|\vec{v}_a - \vec{v}_b\right| = 4\left[(p_a.p_b)^2 - m_a^2 m_b^2\right] = 4\left|\vec{p}_a^{\,*}\right|\sqrt{s}$
  where the Mandelstam variable *s* has been used $s = \left(p_a + p_b\right)^2$

- 2-bodies phase space :

$$dQ_2 = \int_4 (2\pi)^4 \delta^4(p_f - p_i) \frac{d^3 p_1}{(2\pi)^3 2E_1} \frac{d^3 p_2}{(2\pi)^3 2E_2}$$

$$= \frac{1}{16\pi^2} \frac{\left|\vec{p}_1\right|^3 d\Omega_1}{\left|\vec{p}_1\right|^2 E_T - \vec{p}_1.\vec{p}_T E_1} \text{ with } (p_T) = (p_a + p_b)$$

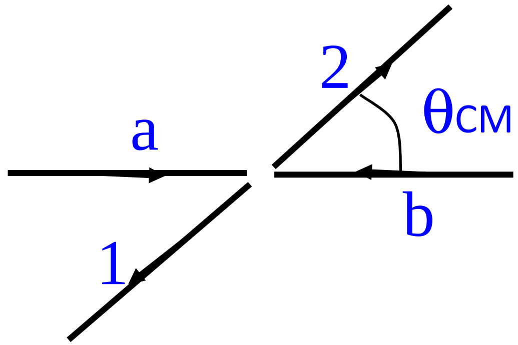
$$= \frac{1}{16\pi^2} \frac{\left|\vec{p}_1^{\,*}\right| d\Omega_1}{\sqrt{s}}$$

1-3 Cross-section

Application : Rutherford scattering formula (2+2 process => n=2).

- Transition rate :

$$\left|T_{fi}\right|^2 = \left(\frac{e^2}{q^2}\right)^2 (p_a + p_1).(p_b + p_2)$$



- Mandelstam variables :

$$s = (p_a + p_b)^2 \quad t = (p_a - p_1)^2 \quad u = (p_a - p_2)^2$$

$$\left|T_{fi}\right|^2 = \left(\frac{4\pi\alpha}{t}\right)^2 (s-u)^2 \text{ and } \frac{d\sigma}{d\Omega} = \frac{1}{64\pi^2 s}\frac{\left|\vec{p}_1^{\,*}\right|}{\left|\vec{p}_a^{\,*}\right|}\left|T_{fi}\right|^2$$

- In the LAB frame (target at rest)

1-3 Cross-section

Application : Rutherford scattering formula (2+2 process => n=2).

- "Classical" case : *a* is a light particle (negligible *m*), *b* at rest initially => use of the LAB frame

$$p_a \simeq \left(|\vec{k}|, \vec{k}\right) \quad p_b \simeq \left(M, \vec{0}\right)$$

$$t \simeq -4|\vec{k}|^2 \sin^2(\theta/2) \quad (s-u)^2 = 16|\vec{k}|^2 M^2$$

$$4\left[(p_a.p_b)^2 - m^2 M^2\right] \simeq 4|\vec{k}|^2 M^2$$

- With the fine structure constant $\alpha = e^2 / 4\pi$

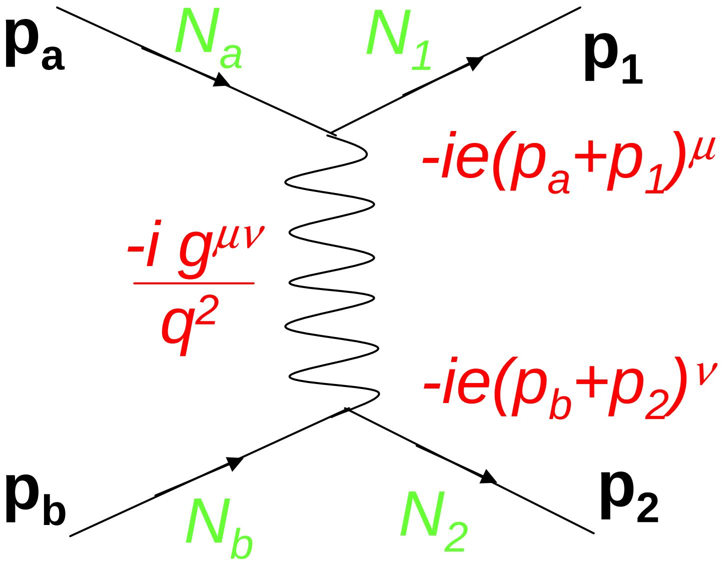
  one gets the standard Rutherford formula :

$$\frac{d\sigma}{d\Omega} = \frac{\alpha^2}{4|\vec{k}|^2 \sin^4(\theta/2)}$$

2- Feynman rules for QED

Summary : “reading” a Feynman diagram

- The basic elements of a diagram are :
  - the external lines
  - the vertex operator
  - the internal lines or propagators



with the transferred momentum : $q = (p_a - p_1) = (p_2 - p_b)$

2- Feynman rules for QED

Including fermions into the game: same procedure

$\Rightarrow$ matrix element reduced expression

$$T_{fi} = -i \int d^4x \, J_\mu^{(a1)} \frac{-g^{\mu\nu}}{q^2} J_\nu^{(b2)}$$

$\Rightarrow$ with a current given by (see lecture on Dirac's equation) :

$$J_\mu^{(fi)} = -e\left(\overline{\psi}_f \gamma_\mu \psi_i\right) = -e\left(\overline{u}_f \gamma_\mu u_i\right) e^{i(p_f - p_i).x}$$

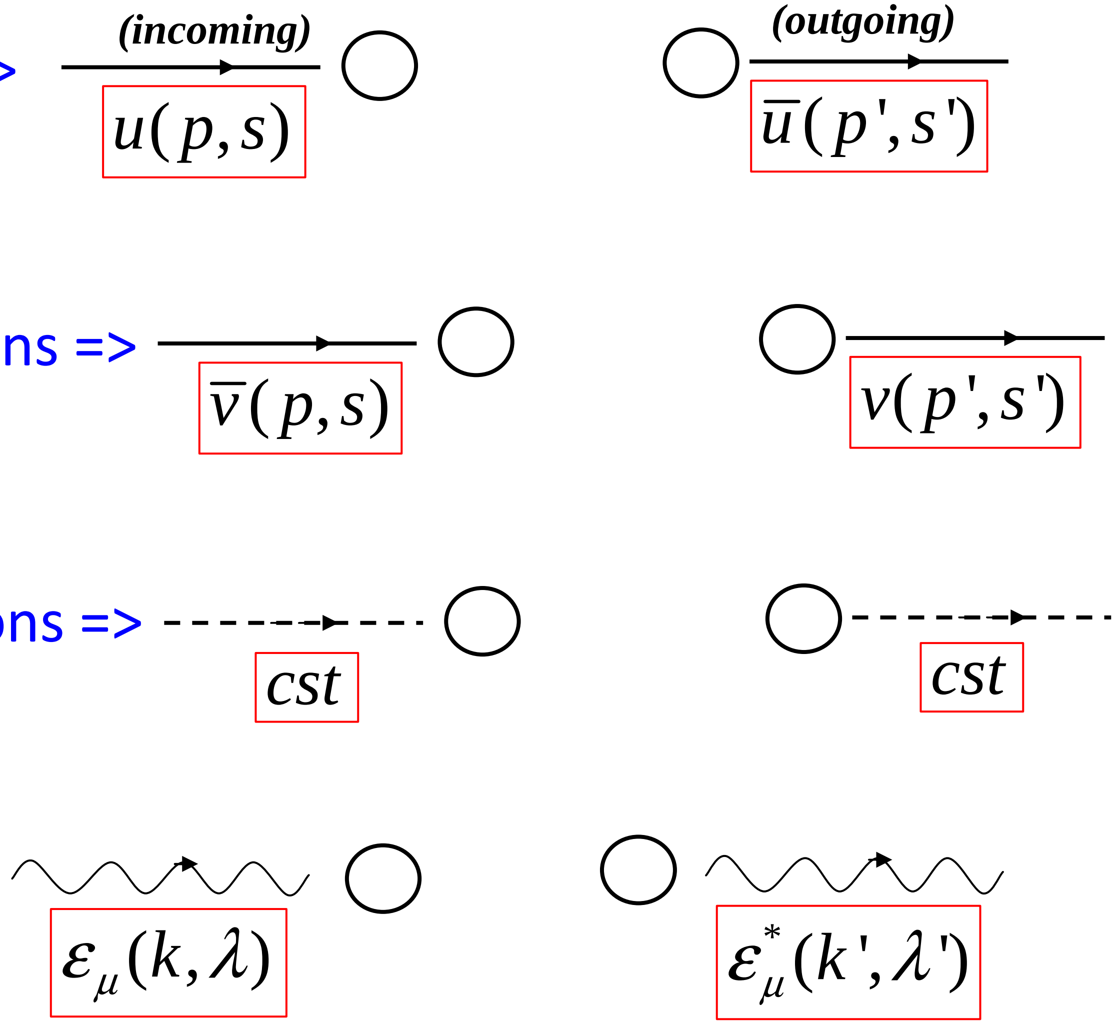
$\Rightarrow$ N.B. to be compared with the spinless case :

$$J_\mu^{(fi)} = -e\left(p_f + p_i\right)_\mu e^{i(p_f - p_i).x}$$

2- Feynman rules for QED

Feynman rules for QED:

- External lines:
  - fermions =>
  - anti-fermions =>
  - spin-0 bosons =>
  - photons =>

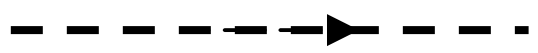


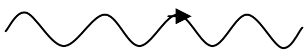
2- Feynman rules for QED

Feynman rules for QED:

- Propagators:

  - fermions => 
  $$\frac{i}{\not{p}-m}=\frac{i(\not{p}+m)}{p^2-m^2}$$

  - spin-0 bosons => 
  $$\frac{i}{p^2-m^2}$$

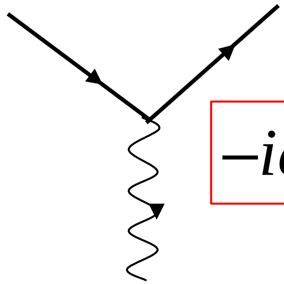
  - photons => 
  $$\frac{-ig^{\mu\nu}}{p^2}$$

2- Feynman rules for QED

Feynman rules for QED:

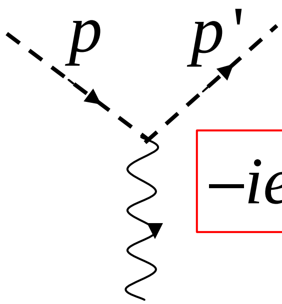
- Vertices :

  - fermions – photon =>



$$-ie\gamma^{\mu}$$

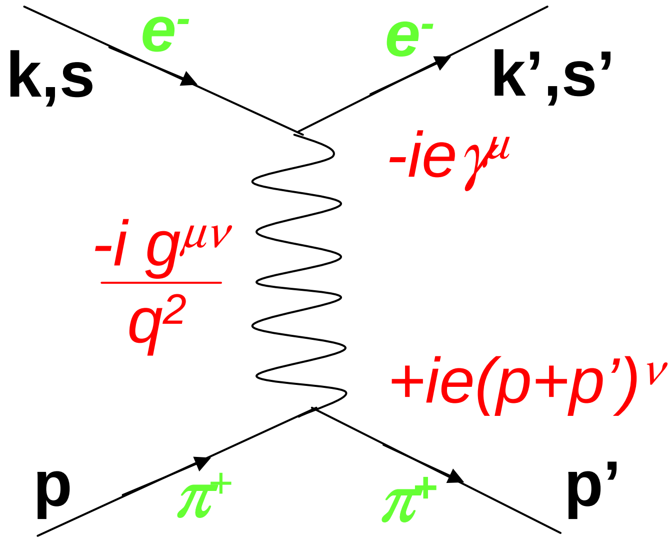
  - spin-0 bosons – photon =>



$$-ie(p+p')^{\mu}$$

3-1 Basic process

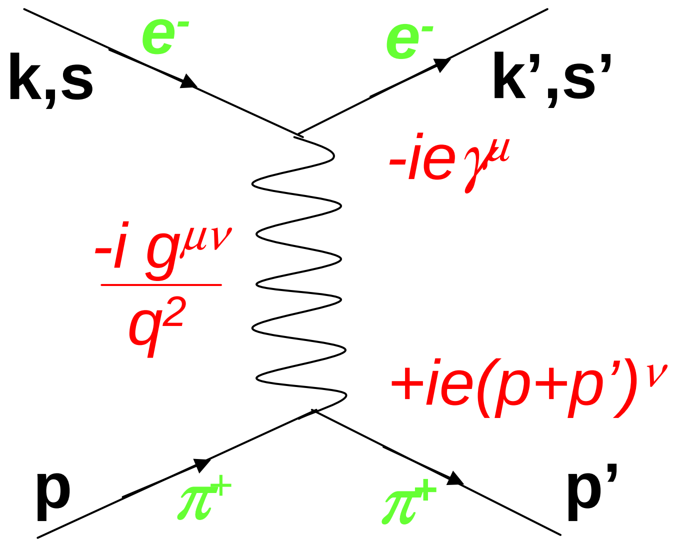
Applications of the Feynman rules to the $e^- \pi^+$ elastic scattering



$$d\sigma = \frac{1}{2E_k 2E_p \left|\vec{v}_k - \vec{v}_p\right|} \int (2\pi)^4 \delta^4(k' + p' - k - p) \overline{\left|T_{fi}\right|^2} \frac{d^3 k'}{(2\pi)^3 2E_{k'}} \frac{d^3 p'}{(2\pi)^3 2E_{p'}}$$

3-1 Basic process

Expression of the transition amplitude :

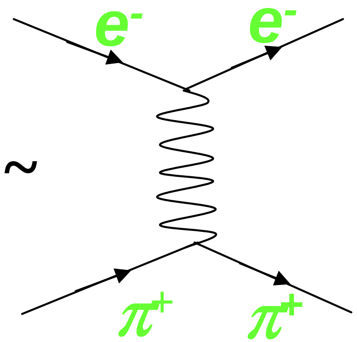


$$T = \bar{u}(k',s')(-ie\gamma^{\mu})u(k,s)\frac{-ig_{\mu\nu}}{q^2}(+ie)(p+p')^{\nu}$$

$$= \frac{-ie^2}{q^2}\bar{u}(k',s')\gamma^{\mu}u(k,s)(p+p')_{\mu}$$

3-1 Basic process

Expression of the |transition amplitude|^2 :

$\mathbf{T}^2 \sim$ |  |^2

$$T^2 = \left(\frac{e^2}{q^2}\right)^2 \left[\overline{u}(k',s')\gamma^{\nu} u(k,s)(p+p')_{\nu}\right]\left[\overline{u}(k',s')\gamma^{\mu} u(k,s)(p+p')_{\mu}\right]^*$$

with the conjugate expression reading :

$$\left[\overline{u}(k',s')\gamma^{\mu} u(k,s)(p+p')_{\mu}\right]^* = \left[\overline{u}(k',s')\gamma^{\mu} u(k,s)\right]^{\dagger}(p+p')_{\mu}$$

$$\left[\overline{u}(k',s')\gamma^{\mu} u(k,s)\right]^{\dagger} = u(k,s)^{\dagger}\gamma^{\mu\dagger}(u^{\dagger}\gamma^0)(k',s')^{\dagger}$$

$$= u(k,s)^{\dagger}\gamma^0\gamma^{\mu}\gamma^0\gamma^0 u(k',s')$$

$$= \overline{u}(k,s)\gamma^{\mu} u(k',s')$$

3-2 Trace techniques for spin $\sum$

Warning : most of the time the simplest measurements use unpolarized electrons (beam of spin-↑ and spin-↓ electrons) and do not include the information on the polarization.

The unpolarized cross-section is defined by :

$$d\bar{\sigma} \equiv \frac{1}{2s+1}\left(d\sigma_{\uparrow\uparrow} + d\sigma_{\uparrow\downarrow} + d\sigma_{\downarrow\uparrow} + d\sigma_{\downarrow\downarrow}\right) = \frac{1}{2}\sum_{s}\sum_{s'} d\sigma_{ss'}$$

This apparent extra complexity leads to an over simplification of the computation thanks to the 'trace technique'.

3-2 Trace techniques for spin $\sum$

Expression of the spin-averaged |transition amplitude|^2 :

$$\left|T^2\right| = \left(\frac{e^2}{q^2}\right)^2 \frac{1}{2}\sum_s \sum_{s'} \left[\bar{u}(k',s')\gamma^\mu u(k,s)\right]\left[\bar{u}(k,s)\gamma^\nu u(k',s')\right]$$

$$\times (p+p')_\mu (p+p')_\nu$$

$$\equiv \left(\frac{e^2}{q^2}\right)^2 L^{\mu\nu} P_{\mu\nu}$$

Definition of leptonic and hadronic (here pionic) tensors.

The hadronic tensor is simply : $P_{\mu\nu} = (p+p')_\mu (p+p')_\nu$

3-2 Trace techniques for spin $\sum$

The complete expression of the leptonic tensor reads :

$$L^{\mu\nu} = \frac{1}{2}\sum_{ss'}\sum_{\alpha\beta\delta\varepsilon}\overline{u}_\alpha(k',s')(\gamma^\mu)_{\alpha\beta}u_\beta(k,s)\overline{u}_\delta(k,s)(\gamma^\nu)_{\delta\varepsilon}u_\varepsilon(k',s')$$

$$= \frac{1}{2}\sum_{\alpha\beta\delta\varepsilon}\underbrace{\sum_{s'}u_\varepsilon(k',s')\overline{u}_\alpha(k',s')}_{(\not{k}'+m_e)_{\varepsilon\alpha}}(\gamma^\mu)_{\alpha\beta}\underbrace{\sum_{s}u_\beta(k,s)\overline{u}_\delta(k,s)}_{(\not{k}+m_e)_{\beta\delta}}(\gamma^\nu)_{\delta\varepsilon}$$

$$= \frac{1}{2}\sum_{\alpha\beta\delta\varepsilon}(\not{k}'+m_e)_{\varepsilon\alpha}(\gamma^\mu)_{\alpha\beta}(\not{k}+m_e)_{\beta\delta}(\gamma^\nu)_{\delta\varepsilon}$$

$$= \frac{1}{2}\mathrm{Tr}\left[(\not{k}'+m_e)(\gamma^\mu)(\not{k}+m_e)(\gamma^\nu)\right]$$

3-2 Trace techniques for spin $\sum$

Trace theorems :

$$\mathrm{Tr}\left[1_{4\times 4}\right] = 4$$

$$\mathrm{Tr}\left[\text{nbr impair de } \gamma\right] = 0$$

$$\mathrm{Tr}\left[\gamma^{\mu}\gamma^{\nu}\right] = 4\left(g^{\mu\nu}\right)$$

$$\mathrm{Tr}\left[\gamma^{\alpha}\gamma^{\mu}\gamma^{\beta}\gamma^{\nu}\right] = 4\left(g^{\alpha\mu}g^{\beta\nu} - g^{\alpha\beta}g^{\mu\nu} + g^{\alpha\nu}g^{\beta\mu}\right)$$

$$\mathrm{Tr}\left[\gamma^{5}\right] = 0$$

$$\mathrm{Tr}\left[\gamma^{5}\gamma^{\alpha}\gamma^{\mu}\right] = 0$$

$$\mathrm{Tr}\left[\gamma^{5}\gamma^{\alpha}\gamma^{\mu}\gamma^{\beta}\gamma^{\nu}\right] = 4i\varepsilon_{\eta\rho\sigma\tau}g^{\eta\alpha}g^{\rho\mu}g^{\sigma\beta}g^{\tau\nu}$$

3-3 Expression of the cross-section

Leptonic tensor (cont'd) :

$$L^{\mu\nu} = \frac{1}{2}\mathrm{Tr}\left[(k\!\!\!/\,'+m_e)(\gamma^\mu)(k\!\!\!/+m_e)(\gamma^\nu)\right]$$

$$= k'_\alpha k_\beta \frac{1}{2}\mathrm{Tr}\left[\gamma^\alpha\gamma^\mu\gamma^\beta\gamma^\nu\right] + m_e^2\frac{1}{2}\mathrm{Tr}\left[\gamma^\mu\gamma^\nu\right]$$

$$= \frac{1}{2}4k'_\alpha k_\beta\left(g^{\alpha\mu}g^{\beta\nu} - g^{\alpha\beta}g^{\mu\nu} + g^{\alpha\nu}g^{\beta\mu}\right) + m_e^2\frac{1}{2}4g^{\mu\nu}$$

$$= 2\left(k'^\mu k^\nu - (k'.k)g^{\mu\nu} + k^\mu k'^\nu\right) + 2m_e^2 g^{\mu\nu}$$

Transferred momentum : $q = k - k' = p' - p$

$$\Rightarrow q^2 = k^2 + k'^2 - 2k.k' = 2(m^2 - k.k')$$

Therefore : $L^{\mu\nu} = \left[2\left(k'^\mu k^\nu + k^\mu k'^\nu\right) + q^2 g^{\mu\nu}\right]$

3-3 Expression of the cross-section

Finally : $L^{\mu\nu}P_{\mu\nu} = \left[2\left(k'^{\mu}k^{\nu} + k^{\mu}k'^{\nu}\right) + q^2 g^{\mu\nu}\right](p+p')_{\mu}(p+p')_{\nu}$

$$\text{with } q_{\mu}L^{\mu\nu} = q_{\nu}L^{\mu\nu} = 0$$

$$L^{\mu\nu}P_{\mu\nu} = \left[2\left(k'^{\mu}k^{\nu} + k^{\mu}k'^{\nu}\right) + q^2 g^{\mu\nu}\right](2p)_{\mu}(2p)_{\nu}$$

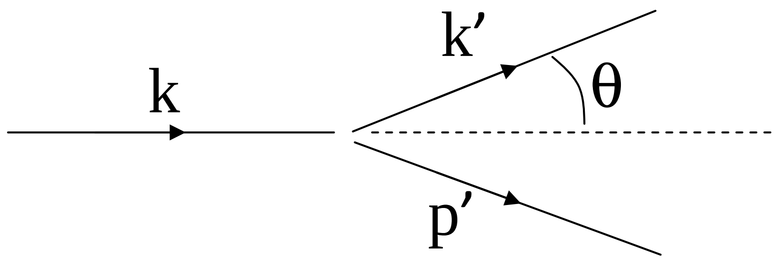
$$= 8\left[2(k'.p)(k.p)\right] + 4m_{\pi}^2 q^2$$

And the cross-section reads :

$$d\sigma = \frac{1}{2E_k 2E_p \left|\vec{v}_k - \vec{v}_p\right|}\left(\frac{4\pi\alpha}{q^2}\right)^2 \left(8\left[2(k'.p)(k.p)\right] + 4m_{\pi}^2 q^2\right) d\Phi_2$$

In the 'laboratory' frame

(pion at rest initially) :



3-3 Expression of the cross-section

The phase space is given by :

$$d\Phi_2 = \frac{1}{16\pi^2}\frac{\left|\vec{k}'\right|^3 d\Omega}{\left|\vec{k}'\right|^2 E_T - \vec{k}'.\vec{k}E_{k'}} \quad \text{with } E_T = m_\pi + E_k,\ E_k \simeq \left|\vec{k}\right|,\ E_{k'} \simeq \left|\vec{k}'\right|$$

$$= \frac{1}{16\pi^2}\frac{E_{k'} d\Omega}{m_\pi + E_k - E_k \cos\theta}$$

$$= \frac{1}{16\pi^2}\frac{E_{k'}}{m_\pi}\frac{d\Omega}{1 + 2\sin^2(\theta/2)\frac{E_k}{m_\pi}}$$

$$\frac{\left|\vec{k}\right|}{\left|\vec{k}'\right|} = \frac{E_k}{E_{k'}}$$

Elastic scattering condition

$$= \frac{1}{16\pi^2}\frac{E_{k'}^2}{m_\pi E_k}$$

where the electron mass has been neglected

$$\Rightarrow q^2 = 2(m^2 - k.k') \simeq -4E_k E_{k'} \sin^2\theta/2$$

3-3 Expression of the cross-section

The cross-section final expression is :

$$\frac{d\sigma}{d\Omega}=\frac{1}{4m_\pi E_k}\left(\frac{4\pi\alpha}{-4E_{k'}E_k\sin^2\theta/2}\right)^2 16(E_{k'}E_k)m_\pi^2\left(1-\sin^2\theta/2\right)$$

$$\times\frac{1}{16\pi^2}\frac{E_{k'}^2}{m_\pi E_k}$$

$$\frac{d\sigma}{d\Omega}=\frac{\alpha^2}{4E_k^2\sin^4\theta/2}\left(\cos^2\theta/2\right)\frac{E_{k'}}{E_k}\equiv\left(\frac{d\sigma}{d\Omega}\right)_{ns}$$

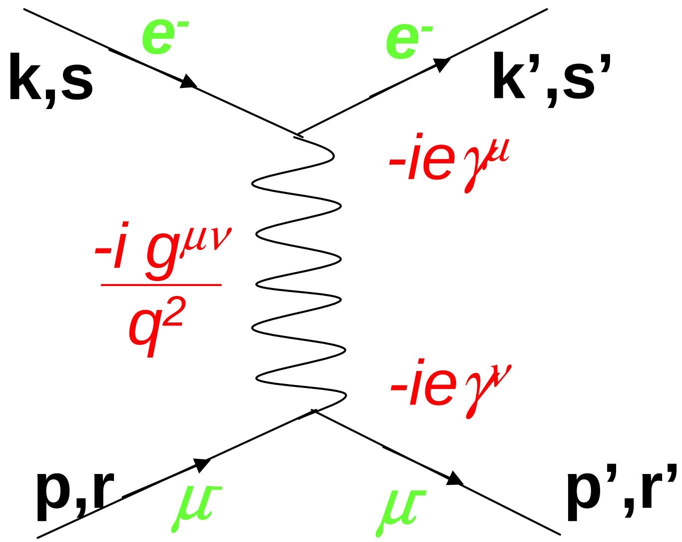
N.B. for a real pion one has to take into account the form factor :

$$-ie(p+p')^\mu\rightarrow -ieF(q^2)(p+p')^\mu$$

the form of the e.m. current is deduced from the Lorentz invariance requirements

3-4 2nd process : e-μ scattering

Consider now a process involving 4 fermions in the initial-final states (i.e. 4 spin indexes) :



The computation procedure is identical to what was done before :

- $T = \bar{u}(k',s')(-ie\gamma^{\mu})u(k,s)\dfrac{-ig_{\mu\nu}}{q^2}\bar{u}(p',r')(-ie\gamma^{\nu})u(p,r)$

- $\left|\bar{T}\right|^2 = \dfrac{1}{(2s_e+1)(2s_\mu+1)}\sum_{rr'ss'} TT^* = \dfrac{1}{4}\sum_{rr'ss'} TT^*$

3-4 2nd process : e-μ scattering

Expression of the average transition amplitude in terms of tensor product:

$$|T|^2 = \left(\frac{e^2}{q^2}\right)^2 \frac{1}{4}\sum_{rr'ss'}\left[\bar{u}(k',s')\gamma^{\mu}u(k,s)\bar{u}(p',r')\gamma_{\mu}u(p,r)\right]$$

$$\times\left[\bar{u}(k,s)\gamma^{\nu}u(k',s')\bar{u}(p,r)\gamma_{\nu}u(p',r')\right]$$

$$= \left(\frac{e^2}{q^2}\right)^2 \frac{1}{2}\sum_{ss'}\left[\bar{u}(k',s')\gamma^{\mu}u(k,s)\bar{u}(k,s)\gamma^{\nu}u(k',s')\right]$$

$$\times\frac{1}{2}\sum_{rr'}\left[\bar{u}(p',r')\gamma_{\mu}u(p,r)\bar{u}(p,r)\gamma_{\nu}u(p',r')\right]$$

$$= \left(\frac{e^2}{q^2}\right)^2 \frac{1}{2}Tr\left[(\not{k}'+m_e)\gamma^{\mu}(\not{k}+m_e)\gamma^{\nu}\right]\times\frac{1}{2}Tr\left[(\not{p}'+m_{\mu})\gamma_{\mu}(\not{p}+m_{\mu})\gamma_{\nu}\right]$$

$$= \left(\frac{e^2}{q^2}\right)^2 L^{\mu\nu}(e)L_{\mu\nu}(\mu)$$

3-4 2nd process : e-μ scattering

The leptonic tensors have the form computed previously :

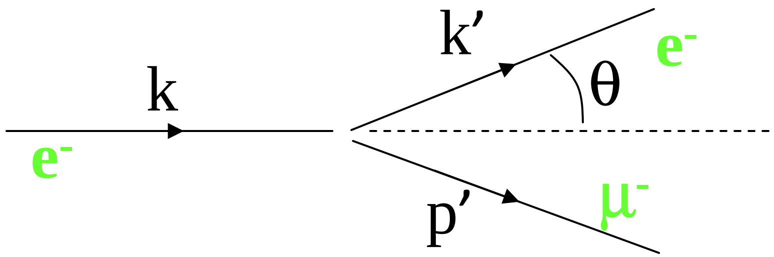
$$L^{\mu\nu}(e) = \frac{1}{2}\mathrm{Tr}\left[(\not{k}' + m_e)(\gamma^\mu)(\not{k} + m_e)(\gamma^\nu)\right]$$

$$= 2\left(k'^\mu k^\nu - (k'.k)g^{\mu\nu} + k^\mu k'^\nu\right) + 2m_e^2 g^{\mu\nu}$$

$$= 2\left(k'^\mu k^\nu + k^\mu k'^\nu + (q^2/2)g^{\mu\nu}\right)$$

$$L^{\mu\nu}(\mu) = 2\left(p'^\mu p^\nu + p^\mu p'^\nu + (q^2/2)g^{\mu\nu}\right)$$

The tensor product is expressed in the 'lab' frame defined with the initial muon at rest and with the further assumption that the electron mass is negligible (same conventions as before).



3-4 2nd process : e-μ scattering

Preliminary remark :

$$q_\mu L^{\mu\nu} = q_\nu L^{\mu\nu} = 0 \Rightarrow L^{\mu\nu}(e)L_{\mu\nu}(\mu) = L^{\mu\nu}(e)L^{eff}_{\mu\nu}(\mu)$$

$$\text{where } L^{eff}_{\mu\nu}(\mu) = 2\left(2p_\mu p_\nu + (q^2/2)g_{\mu\nu}\right)$$

Therefore :

$$L^{\mu\nu}(e)L_{\mu\nu}(\mu) = 4\left(k'^\mu k^\nu + k^\mu k'^\nu + (q^2/2)g^{\mu\nu}\right)\left(2p_\mu p_\nu + (q^2/2)g_{\mu\nu}\right)$$

$$= 4(4(p.k)(p.k') + (q^2/2)(2(k.k') + 2p^2) + 4(q^2/2)^2)$$

$$= 4(4(p.k)(p.k') + (q^2/2)(-q^2 + 2p^2) + 4(q^2/2)^2)$$

$$= 4(4(p.k)(p.k') + q^2p^2 + (q^2)^2/2)$$

Where we used : $q^2 \simeq -2k.k' \simeq -4E_k E_{k'} \sin^2\frac{\theta}{2}$

3-4 2nd process : e-μ scattering

Therefore :

$$L^{\mu\nu}(e)L_{\mu\nu}(\mu) = 4(4(p.k)(p.k') + q^2 p^2 + (q^2)^2/2)$$

$$= 4\left(4m_\mu^2 E_k E_{k'} + 4(-E_k E_{k'} \sin^2 \tfrac{\theta}{2}) m_\mu^2 - 2E_k E_{k'} \sin^2 \tfrac{\theta}{2} q^2\right)$$

$$= 16 m_\mu^2 E_k E_{k'} (1 - \sin^2 \tfrac{\theta}{2} - \sin^2 \tfrac{\theta}{2} \frac{q^2}{2m_\mu^2})$$

$$= 16 m_\mu^2 E_k E_{k'} (\cos^2 \tfrac{\theta}{2} - \sin^2 \tfrac{\theta}{2} \frac{q^2}{2m_\mu^2})$$

And

$$\frac{d\sigma}{d\Omega} = \underbrace{\frac{\alpha^2}{4E_k^2 \sin^4 \frac{\theta}{2}} \left(\cos^2 \tfrac{\theta}{2}\right) \frac{E_{k'}}{E_k}}_{\left(\frac{d\sigma}{d\Omega}\right)_{ns}} \times (1 - \tan^2 \tfrac{\theta}{2} \frac{q^2}{2m_\mu^2})$$

3-4 2nd process : e-μ scattering

Remarks :

- The term $\left(\frac{d\sigma}{d\Omega}\right)_{ns}$ represents the scattering on a spin-less, structure-less particle (e.g. electron-pion w/o F.F.)
- There is an extra contribution $\propto \tan^2\frac{\theta}{2}\frac{q^2}{2m_\mu^2}$ due to the existence of the spin => coupling to the charge <u>and</u> the magnetic moment
- The above computation is transposable to the 'annihilation' case : $e^+ + e^- \to \mu^+ + \mu^-$

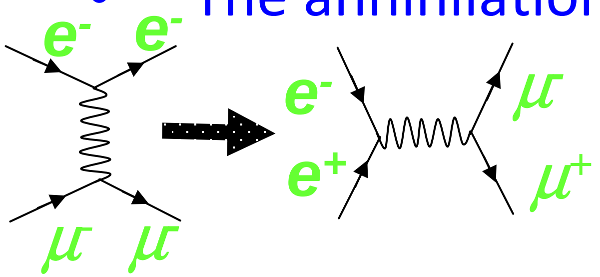
3-4 2nd process : e-μ scattering

Remarks :

- In the CM frame (neglecting all leptons masses) one can express simply for the process $e^- + \mu^- \rightarrow e^- + \mu^-$:

$$|T|^2 = 2e^4 \frac{s^2 + u^2}{t^2} \text{ with } \begin{cases} s = (k+p)^2 \\ t = (k-k')^2 \\ u = (k-p')^2 \end{cases}$$

- The annihilation process is obtained by substituting :



$$k' \rightarrow -p \Rightarrow s \rightarrow t \Rightarrow |T|^2 = 2e^4 \frac{t^2 + u^2}{s^2}$$

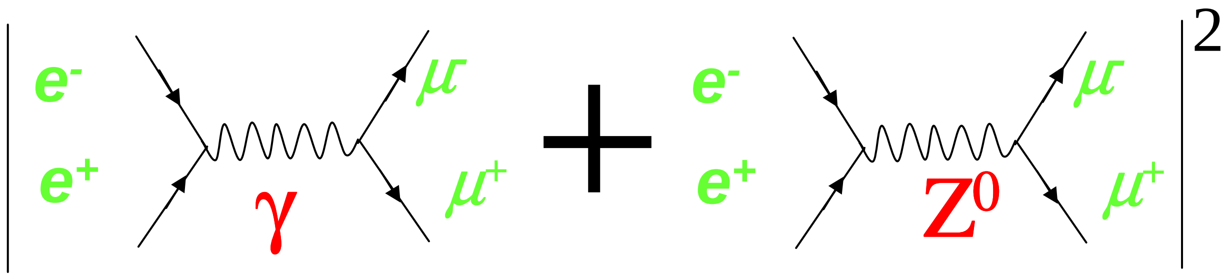
$$\left(\frac{\partial \sigma}{\partial \Omega}\right)_{CM} = \frac{\alpha^2}{4s}\left(1 + \cos^2\theta\right) \text{ and } \sigma = \frac{4\pi\alpha^2}{3s}$$

3-4 2^{nd} process : e-μ scattering

Remarks :

- Good agreement with experimental data (angular distribution and total cross-sections) at lower energies

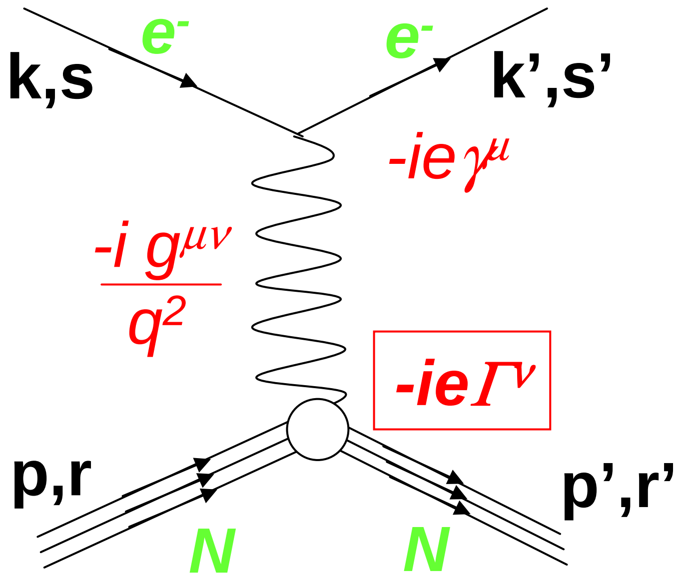
- At higher energies one feels the effects of the weak interaction
  $\sqrt{s} \sim 90\ \text{GeV}$



3-5 Electron-Nucleon scattering

General features :

- The electron-muon process is a prototype to address the problem of electron scattering over non point-like particles such as nucleons (***uud*** or ***udd***)



- What is the form of the generalized vertex function?

3-5 Electron-Nucleon scattering

- One assumes a general form of the current :

$$J^\mu = \bar{u}(p')\Gamma^\mu u(p) = \bar{u}(p')\left(A(q^2)\gamma^\mu + B(q^2)p^\mu + C(q^2)p'^\mu\right)u(p)$$

requesting the current conservation and the Lorentz invariance

- Using the Gordon identity :

$$\bar{u}(p')\left(\gamma^\mu\right)u(p) = \frac{1}{2M}\bar{u}(p')\left((p+p')^\mu + i\sigma^{\mu\nu}q_\nu\right)u(p)$$

one gets an equivalent parametrization :

$$J^\mu = -ie\bar{u}(p')\left(F_1(q^2)\gamma^\mu + F_2(q^2)\frac{i\kappa\sigma^{\mu\nu}q_\nu}{2M} + F_3(q^2)q^\mu\right)u(p)$$

- ***$F_i(q^2)$*** are the nucleon form factors (in fact ***$F_3(q^2)$=0***)

3-5 Electron-Nucleon scattering

The cross-section reads :

$$\frac{\partial\sigma}{\partial\Omega}=\left(\frac{\partial\sigma}{\partial\Omega}\right)_{ns}\left(\underbrace{\left(F_1^{\,2}-\frac{\kappa q^2}{4M^2}F_2^{\,2}\right)}_{G_E(q^2)}-\underbrace{\left(F_1^{\,2}+\kappa F_2^{\,2}\right)}_{G_M(q^2)}\frac{q^2}{2M^2}\tan^2\theta/2\right)$$

- G_E : "electric" from factor
- G_M : "magnetic" form factor

- N.B. individual contributions of $e^+ + e^- \rightarrow q+\overline{q}$ computable as electron-muon scatterings $e \rightarrow e_q$