Chapter 3

Relativistic propagation equations

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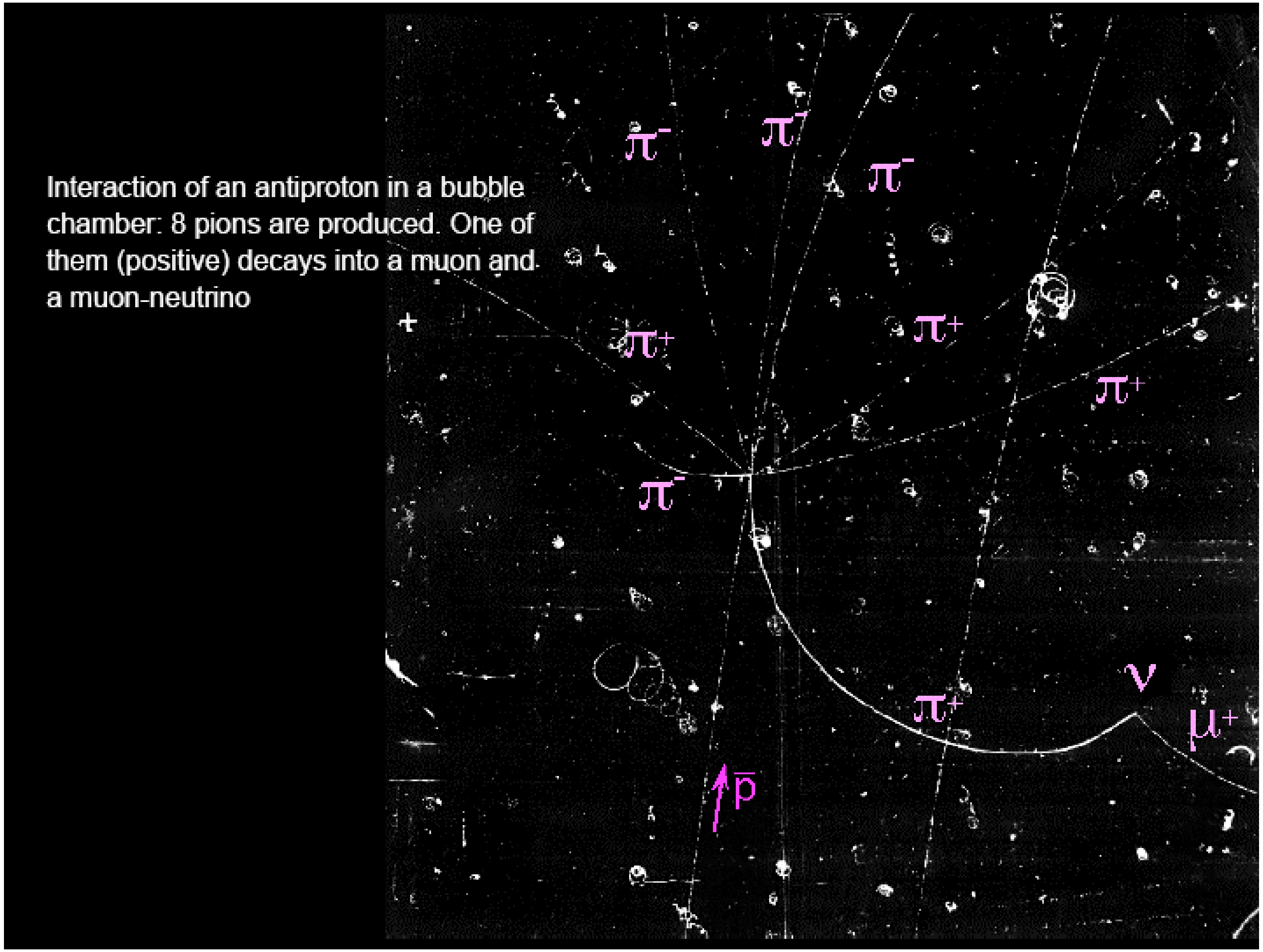
1- Antiparticles and relativity

Observation of antiparticles :

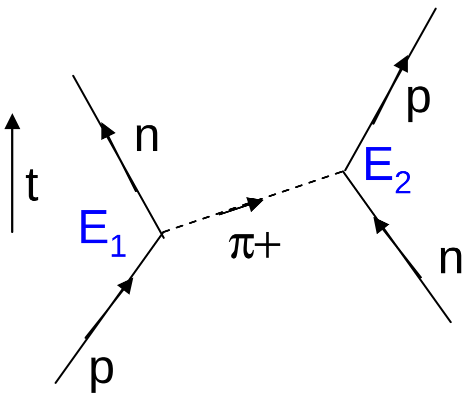


1- Antiparticles and relativity

Historically:

- Predication of the positron by Dirac (1928)
- Experimental signature by Anderson (1932)
- Theoretical difficulties : "negative energies" ➔ holes theory
- Matter-antimatter asymmetry

Studying charge-exchange diagram with some basics of 4-vectors:



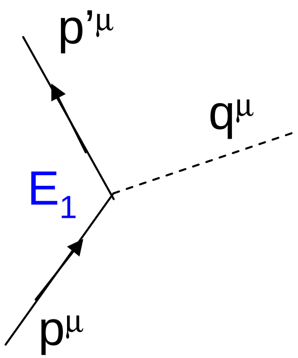
$$p^{\mu} = (p^0 = E = \gamma M, \vec{p} = \gamma M \vec{v})$$

$$\gamma = 1 \Big/ \sqrt{1-\beta^2} \quad \text{and} \quad \beta = v / c$$

$$p^2 = p^{\mu} p_{\mu} = E^2 - \vec{p}^{\,2} = E^2(1-v^2) = M^2$$

1- Antiparticles and relativity

The charge exchange process is forbidden classically



$$p^\mu = p'^\mu + q^\mu$$

$$q^2 = (p^\mu - p'^\mu)^2 = 2M^2 - 2p^\mu p'_\mu$$

$$= 2M^2\left(1 - \frac{1 - \vec{v}.\vec{v}}{\sqrt{(1-v^2)(1-v'^2)}}\right)$$

$$\leq 0 \text{ but } q^2 = m^2 ?$$

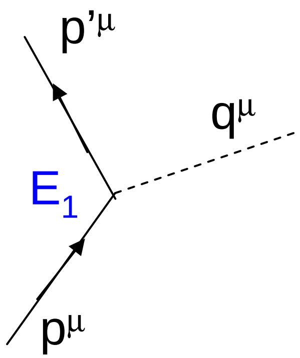
- Between E_1 and E_2 a virtual particle state may be exchanged for a time less than $\Delta t \leq \frac{\hbar}{m}$
- "E1 – E2" : space-like interval between the 2 events

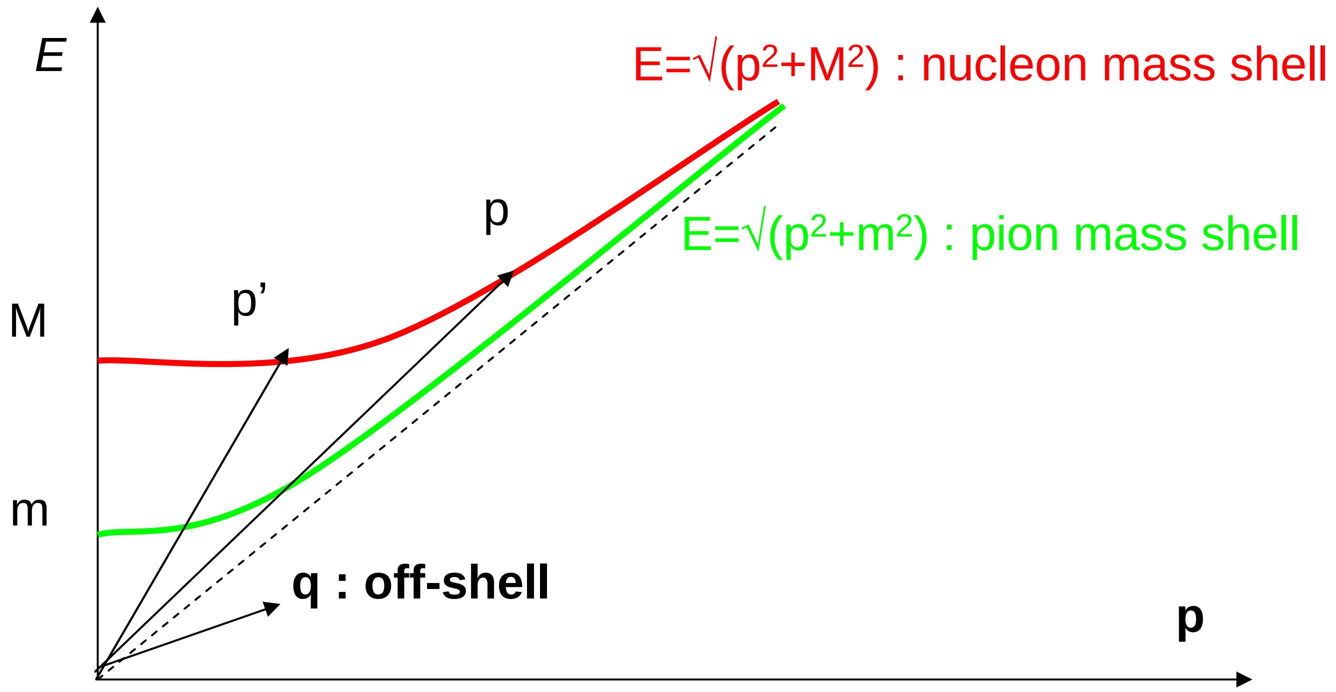
$$q^2 = q_0^2 - \vec{q}^2 = q_0^2\left(1 - V^2\right) \text{ with } V: \text{ pion velocity}$$

$$\Rightarrow (\Delta t)^2 - (\Delta \vec{x})^2 = (\Delta t)^2\left(1 - V^2\right) < 0$$

1- Antiparticles and relativity

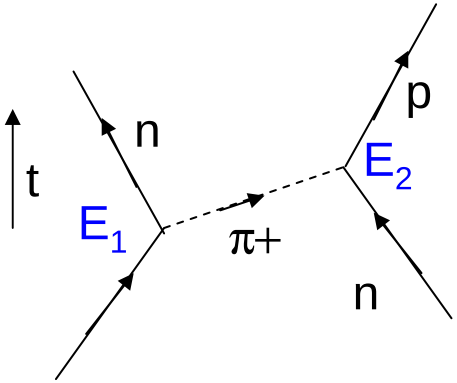
Off-shell virtual particle exchange :



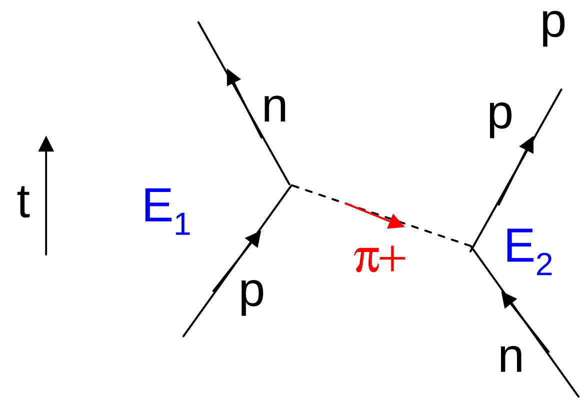


1- Antiparticles and relativity

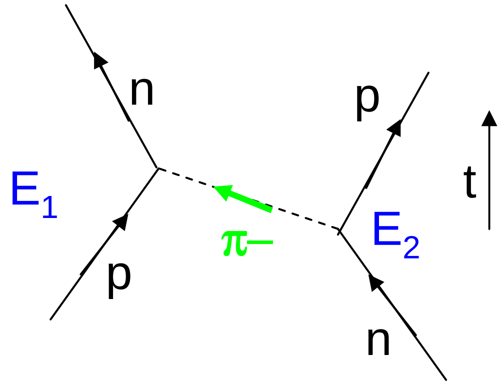
By reversing time order (E2 observed before E1) the π^+ should have been absorbed before emission (causality violation)



E1 before E2



E1 after E2



$\Rightarrow$emission of π^+ antiparticle : π^-

2- Klein-Gordon equation

Introduction :

- Schrödinger equation describes the evolution of a non-relativistic wavefunction using the canonical quantization :

$$\vec{x} \rightarrow \vec{x}$$

$$\vec{p} \rightarrow -i\hbar\vec{\nabla}$$

$$E \rightarrow i\hbar\tfrac{\partial}{\partial t}$$

applied to the energy definition :

$$E = \frac{\vec{p}^{\,2}}{2m} + V(\vec{x})$$

$$\Rightarrow i\hbar\frac{\partial\psi}{\partial t}(\vec{x},t) = \left(\frac{-\hbar^2}{2m}\Delta + V\right)\psi(\vec{x},t)$$

2- Klein-Gordon equation

Introduction :

- Reminder : covariant formalism

$$x^\mu \equiv \left(x^0 = ct, \vec{x}\right) \text{ and } x_\mu \equiv \left(x^0 = ct, -\vec{x}\right)$$

$$\partial_\mu \equiv \frac{\partial}{\partial x^\mu} = \left(\frac{\partial}{c\partial t}, \vec{\nabla}\right) \text{ and } \partial^\mu \equiv \frac{\partial}{\partial x_\mu} = \left(\frac{\partial}{c\partial t}, -\vec{\nabla}\right)$$

- 4-dimensional canonical quantization :

$$p^\mu \equiv \left(p^0 = E, \vec{p}\right) \;\; \rightarrow i\hbar\partial^\mu = \left(i\hbar\frac{\partial}{c\partial t}, -i\hbar\vec{\nabla}\right)$$

- How to derive a relativistic evolution equation? Following the same prescription than in the Schrödinger case but with the relativistic energy definition

$$E^2 = \vec{p}^2 c^2 + m^2 c^4$$

2-1 K-G equation derivation

Preliminary remark : why not starting from $E = \sqrt{\vec{p}^2 c^2 + m^2 c^4}$?

- Time-space asymmetry
- Difficult development of the square root
- But : only positive-energy solutions…

Derivation : canonical quantization applied

$$p^2 = p^\mu p_\mu = p_0^2 - \vec{p}^2 = m^2 c^2$$

$$\Downarrow$$

$$-\hbar^2 \partial^\mu \partial_\mu \psi(x) = m^2 c^2 \psi(x) \text{ with } \partial^\mu \partial_\mu \equiv \Box = \frac{\partial^2}{c^2 \partial t^2} - \Delta$$

Finally : $\boxed{\left(\Box + \frac{m^2 c^2}{\hbar^2} \right) \psi(x) = 0}$

2-1 K-G equation derivation

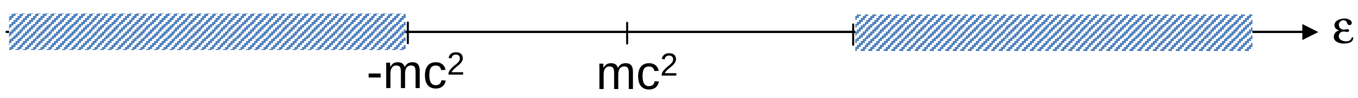
Remarks :

- The photon (boson) is solution of KG's propagation equation (with m=0)
- KG allows to describe all (anti-)particles within the same formalism
- Negative energy solutions? Consider a plane wave

$$\psi(x) = Be^{i(\vec{p}.\vec{x}-\varepsilon t)/\hbar} = Be^{ip^{\mu}x_{\mu}/\hbar}$$

$$\left(\Box + \frac{m^2c^2}{\hbar^2}\right)\psi = 0 \Rightarrow \varepsilon^2 = \vec{p}^2c^2 + m^2c^4$$

$$\varepsilon = \pm E_p \text{ with } E_p = \sqrt{\vec{p}^2c^2 + m^2c^4}$$



2-2 Probabilistic interpretation

Reminder :

- In the non-relativistic case the probabilistic interpretation of wavefunctions reads (continuity equation) :

$$\frac{\partial \rho}{\partial t} + \vec{\nabla}.\vec{J} = 0 \text{ where}$$

$$\begin{cases} \rho = \psi \psi^* \\ \vec{J} = \Re\left(\frac{-i\hbar}{m} \psi^* \vec{\nabla} \psi \right) = \frac{-i\hbar}{2m} \left(\psi^* \vec{\nabla} \psi - \psi \vec{\nabla} \psi^* \right) \end{cases}$$

with the integral condition : $\int d^3x \left| \psi(\vec{x},t) \right|^2 = N = ct$

2-2 Probabilistic interpretation

In the relativistic case the same procedure can be followed :

$$\partial^\mu \partial_\mu \psi(x) + m^2 c^2 \psi(x) \ = 0$$

$$\Rightarrow \begin{cases} \psi^* \partial^\mu \partial_\mu \psi(x) + m^2 c^2 \psi^* \psi(x) \ = 0 \\ \psi \partial^\mu \partial_\mu \psi^*(x) + m^2 c^2 \psi \psi^*(x) \ = 0 \end{cases}$$

$$\Rightarrow \psi^* \partial^\mu \partial_\mu \psi - \psi \partial^\mu \partial_\mu \psi^* = 0$$

$$\Leftrightarrow \partial^\mu J_\mu = 0 \text{ with } J_\mu = \frac{i\hbar}{2m}\left(\psi^* \partial^\mu \psi - \psi \partial^\mu \psi^*\right)$$

with the integral charge conservation condition :

$$\int d^3x \rho(x) = N = ct \ \text{ where } \rho(x) = \frac{i\hbar}{2mc^2}\left(\psi^* \frac{\partial}{\partial t}\psi - \psi \frac{\partial}{\partial t}\psi^*\right)$$

and $\rho \leq$ or ≥ 0

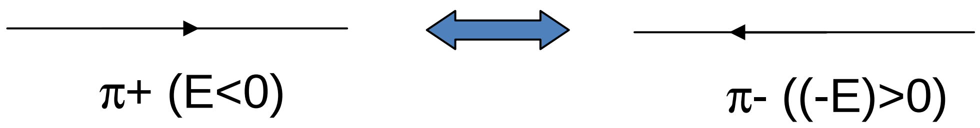
2-2 Probabilistic interpretation

Interpretation? The number of particles is not conserved (possible annihilations) but some charges may be conserved : multiplying ρ by the electric charge in the case of a plane wave for instance:

$$\psi_{\pm}(x) = B_{\pm} e^{i(\vec{p}.\vec{x} \mp E_p t)/\hbar} \Rightarrow \rho_{\pm} = \pm e \frac{E_p}{mc^2} \left| B_{\pm} \right|^2$$

$\rho_{\pm}$ represents the charge density which may be of both signs!

A negative energy particle represents an anti-particle moving in reverse time order!



2-3 Diffusion amplitude

Reminder : transition amplitude (covariant expression) from initial (i) to final *(f)* state on the action of a perturbation potential *V.*

$$T_{fi} = -i\int d^4x\, \psi_f^*(x) V(x) \psi_i(x)$$

For a charged particle moving in an (e.g. electromagnetic) potential *A*$^\mu$: $p^\mu \rightarrow p^\mu + eA^\mu$ ie $i\hbar\partial^\mu \rightarrow i\hbar\partial^\mu + eA^\mu$

The KG equation reads therefore :

$$\left(\partial^\mu \partial_\mu + m^2\right)\psi(x) = -V\psi(x) \text{ where } V = -ie\left(\partial_\mu A^\mu + A^\mu \partial_\mu\right) + o(e^2)$$

2-3 Diffusion amplitude

Amplitude computation :

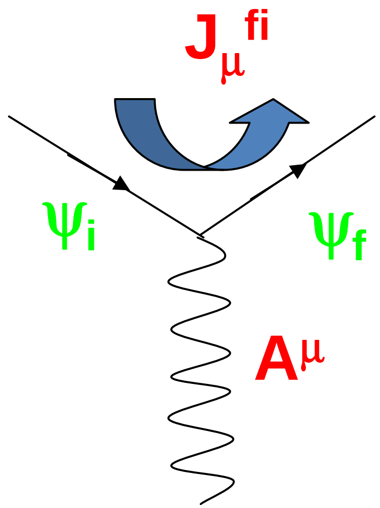
$$T_{fi} = -i\int d^4x\, \psi_f^*(x) V(x) \psi_i(x)$$

$$T_{fi} = -i\int d^4x\, \psi_f^*(x)\left(-ie\left(\partial_\mu A^\mu + A^\mu \partial_\mu\right)\right)\psi_i(x)$$

$$= -i\int d^4x\,(-ie)\left[\psi_f^* A^\mu \partial_\mu \psi_i - \partial_\mu \psi_f^* A^\mu \psi_i\right]$$

$$= -i\int d^4x\, A^\mu (-ie)\left[\psi_f^* \partial_\mu \psi_i - \partial_\mu \psi_f^* \psi_i\right]$$

$$= -i\int d^4x\, A_\mu J_{fi}^\mu$$



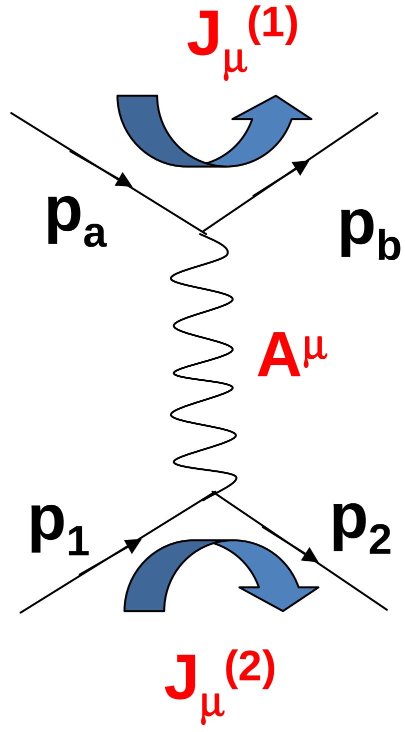
With A^μ linked to its source through $\left(\partial^\nu \partial_\nu\right) A^\mu(x) = J^\mu_{(2)}$

(see later)

2-3 Diffusion amplitude

Starting with Feynman diagrams :

$$T_{fi} = -i\int d^4x\, A_\mu J^\mu_{(1)} = -i\int d^4x\, J_{\mu(2)}\left(\frac{-1}{q^2}\right)J^\mu_{(1)}$$



3- Introduction to gauge theory

- Particle physics relies on quantum field theory which is commonly expressed in Lagrangian formalism.

- Reminder :

  - In classical mechanics the particle's motion is described by the Lagrange equations:

$$\frac{d}{dt}\left(\frac{\partial L}{\partial \dot{q}_i}\right) - \frac{\partial L}{\partial q_i} = 0$$

  where q_i are the generalized coordinates of the particles and $\dot{q}_i = \frac{dq_i}{dt}$ their time derivatives.

  - The Lagrangian of the system is defined as $L = T - V$ where T and V are the kinetic and potential energies.

3-1 Lagrangian formalism

- From discrete to continuous variables $\psi(\vec{x},t)$:

  - the Lagrangian is replaced by a Lagrangian density

$$L(q_i,\dot{q}_i,t) \rightarrow \mathcal{L}(\psi,\partial_\mu\psi,x_\mu)$$

  - the normalization of the Lagrangian density is such that :

$$L = \int d^3x\,\mathcal{L}$$

  - the Euler-Lagrange equations read :

$$\partial_\mu\left(\frac{\partial\mathcal{L}}{\partial(\partial_\mu\psi)}\right) - \frac{\partial\mathcal{L}}{\partial\psi} = 0$$

- Starting from the Lagrangian density one defines an action :

$$S(\psi) = \int d^4x\,\mathcal{L}(\psi,\partial_\mu\psi,x_\mu)$$

3-1 Lagrangian formalism

- Noether's theorem : each invariance of the theory (Lagrangian density) implies the conservation of a charge and a current

- For instance the variation of the action $S' = S(\psi')$ expressed in terms of the transformed fields ψ' under a local transformation depending on the parameter $\alpha(x)$ reads :

$$\delta S = S' - S = \int d^4x\, \alpha(x)\, \partial_\mu J^\mu$$

- The least action principle leads to the continuity equation :

$$\partial_\mu J^\mu = 0$$

describing the conservation of the charge $Q = \int d^3x\, J^0$

3-1 Lagrangian formalism

- The physics of a given type of particle is described through a Lagrangian density involving quantum fields which can be seen as creation/annihilation operators of particles in the standard of 2nd quantization.

- For example the free movement of a spinless particle is described by the following Lagrangian density:

$$\mathcal{L}_{free} = \partial_\mu \psi^\dagger \partial^\mu \psi - m^2 \psi^\dagger \psi$$

Applying the Euler-Lagrange equations $\partial_\mu \left(\frac{\partial \mathcal{L}}{\partial(\partial_\mu \psi^\dagger)} \right) - \frac{\partial \mathcal{L}}{\partial \psi^\dagger} = 0$

we get the K-G equation : $\partial_\mu \left(\partial^\mu \psi \right) - (-m^2 \psi) = 0$

3-2 Gauge invariance

- The invariance of the theory (Lagrangian) under a

  - space translation

  - time translation

  - space rotation

  is associated with the conservation of $\vec{p},\ E,\ \vec{J}$

- Those symmetries are “space-time” like. The theory can be also invariant under internal symmetries. For instance for an electron described by a field ψ the Lagrangian is invariant under the global phase transform :

$$\psi \to e^{i\alpha}\,\psi$$

The transforms $U\left(\alpha\right)=e^{i\alpha}$ constitute the Abelian group $U\left(1\right)$

3-2 Gauge invariance

- The conserved quantity in that case corresponds to the electrical charge.
- Starting from an infinitesimal transform $\psi \rightarrow (1+i\alpha)\psi$
  one derives the real form of the conserved current…
- Physically the existence of a symmetry implies that a quantity is not observable (e.g. the invariance under a space translation means that it is not possible to fix an absolute position in space which can be therefore chosen arbitrarily).
- In the $U(1)$ case the quantity α is called a global gauge.

In the particle physics Standard Model the fundamental interactions are built on symmetry principles, those of the ***local gauge*** transforms.

3-2 Gauge invariance

Local gauge invariance of the free particles Lagrangian :

- under the local transform $\psi \to e^{i\alpha}\,\psi$ where the α parameter depends on x^{μ} the Lagrangian

$$\mathcal{L}_{free} = \partial_{\mu}\psi^{\dagger}\partial^{\mu}\psi - m^2\psi^{\dagger}\psi$$

is not invariant :

$$\psi \to e^{i\alpha}\,\psi \quad \Rightarrow \partial^{\mu}\psi \to \left(\partial^{\mu} + i\left(\partial^{\mu}\alpha\right)\right)\psi$$

$$\psi^{\dagger} \to e^{-i\alpha}\,\psi^{\dagger} \quad \Rightarrow \partial^{\mu}\psi^{\dagger} \to \left(\partial^{\mu} - i\left(\partial^{\mu}\alpha\right)\right)\psi^{\dagger}$$

$$\psi^{\dagger}\psi \to \psi^{\dagger}\psi \quad ☺$$

$$\partial_{\mu}\psi^{\dagger}\partial^{\mu}\psi \to \left(\partial^{\mu} - i\left(\partial^{\mu}\alpha\right)\right)\psi^{\dagger}\left(\partial^{\mu} + i\left(\partial^{\mu}\alpha\right)\right)\psi$$

$$= \partial_{\mu}\psi^{\dagger}\partial^{\mu}\psi - i\left(\partial^{\mu}\alpha\right)\psi^{\dagger}\partial^{\mu}\psi + \psi^{\dagger}i\left(\partial^{\mu}\alpha\right)\psi \ldots \quad ☹$$

3-2 Gauge invariance

Local gauge invariance of the free particles Lagrangian :

- To force the invariance one introduces a covariant derivative :

$$D^{\mu}\psi = \left(\partial^{\mu} - ieA^{\mu}\right)\psi$$

where A^{μ} is the gauge field which should transform as :

$$A^{\mu} \rightarrow A^{\mu} + \frac{1}{e}\partial^{\mu}\alpha$$

in order to balance the transform of the derivative terms

$$D^{\mu}\psi \rightarrow \left(\partial^{\mu} + i\left(\partial^{\mu}\alpha\right) - ieA^{\mu} - ie\left(\frac{1}{e}\partial^{\mu}\alpha\right)\right)\psi$$

$$D^{\mu}\psi^{\dagger} \rightarrow \left(\partial^{\mu} - i\left(\partial^{\mu}\alpha\right) + ieA^{\mu} + ie\left(\frac{1}{e}\partial^{\mu}\alpha\right)\right)\psi^{\dagger}$$

3-2 Gauge invariance

Local gauge invariance of the free particles Lagrangian :

- The Lagrangian is modified into :

$$\begin{aligned}\mathscr{L}_{\text{int}} &= D_\mu\psi^\dagger D^\mu\psi - m^2\psi^\dagger\psi \\ &= \partial_\mu\psi^\dagger\partial^\mu\psi - m^2\psi^\dagger\psi \\ &+ie\left[A_\mu\psi^\dagger\partial^\mu\psi - \partial_\mu\psi^\dagger A^\mu\psi\right] \\ &+e^2 A_\mu\psi^\dagger A^\mu\psi\end{aligned}$$

- Forcing the *U(1)* invariance introduces a new vector field, the gauge field, which couples to the particles through 2 different types of vertices. Generic coupling term: $\boxed{-J_\mu A^\mu}$

- This gauge field is associated to the photon, responsible for the electromagnetic interaction.

3-3 U(1) gauge field

To really associate the gauge field of the U(1) symmetry to the photon it is mandatory to include the dynamics of the photon itself:

- Propagation equation in vacuum : $\Box A^{\mu} = 0$
- Photon interaction with its sources (Maxwell equations) :

$$\boxed{\partial_{\mu} F^{\mu\nu} = J^{\nu}}$$

with the field tensor : $F^{\mu\nu} = \partial^{\mu} A^{\nu} - \partial^{\nu} A^{\mu}$

- Reminder : classical fields

$$A^{\mu} = \left(V, \vec{A}\right)$$

$$F_{0i} = \partial_0 A_i - \partial_i A_0 = -\frac{\partial\left(\vec{A}\right)_i}{\partial t} - \left(\vec{\nabla}\right)_i V = \left(\vec{E}\right)_i$$

$$F_{ij} = \partial_i A_j - \partial_j A_i = \vec{\nabla}_i \vec{A}_j - \vec{\nabla}_j \vec{A}_i = \varepsilon_{ijk}\left(\vec{\nabla} \times \vec{B}\right)_k = \varepsilon_{ijk} \vec{B}_k$$

3-3 U(1) gauge field

- Propagation equations of the interacting field :

$$\partial_\mu F^{\mu\nu} = J^\nu$$

$$\Downarrow$$

$$\partial_\mu \left(\partial^\mu A^\nu - \partial^\nu A^\mu \right) = J^\nu$$

$$\Box A^\nu - \partial^\nu \underbrace{\left(\partial_\mu A^\mu \right)}_{\substack{=0 \\ \text{in Lorentz} \\ \text{gauge}}} = J^\nu \Rightarrow \boxed{\Box A^\nu = J^\nu}$$

- Dynamic term for the photon

$$\mathcal{L}_{free} = -\frac{1}{4} F_{\mu\nu} F^{\mu\nu} \quad \Bigg| \quad \partial_\mu \left(\frac{\partial \mathcal{L}}{\partial \left(\partial_\mu A_\nu \right)} \right) - \frac{\partial \mathcal{L}}{\partial A_\nu} = 0$$

- No mass term (!) : $m^2 A_\mu A^\mu$ not invariant under *U(1)* transform

3-3 U(1) gauge field

Summary :

- Particle physics is described by local gauge theories
- Each invariance is associated with gauge field(s)
- Gauge fields couple to the particles
- For spinless particles the Q.E.D. Lagrangian density reads :

$$\mathcal{L}_{spin-0} = \partial_\mu \psi^\dagger \partial^\mu \psi - m^2 \psi^\dagger \psi$$
$$+ie\left[A_\mu \psi^\dagger \partial^\mu \psi - \partial_\mu \psi^\dagger A^\mu \psi\right] + e^2 A_\mu \psi^\dagger A^\mu \psi$$
$$-\frac{1}{4} F_{\mu\nu} F^{\mu\nu}$$

- What about ½-spin particles description?

4- The Dirac equation

Introduction (non-relativistic case) : the Pauli equation

- Preliminary remark on Pauli matrices :

$$\sigma_1 = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} \quad \sigma_2 = \begin{pmatrix} 0 & -i \\ i & 0 \end{pmatrix} \quad \sigma_3 = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}$$

  if $\vec{O}_1$ and $\vec{O}_2$ are operators commuting with the Pauli matrices then

$$\left(\vec{\sigma}.\vec{O}_1\right)\left(\vec{\sigma}.\vec{O}_2\right) = \vec{O}_1.\vec{O}_2 + i\vec{\sigma}.\left(\vec{O}_1 \times \vec{O}_2\right)$$

- Applying the previous relation to $\vec{P}$ leads to :

$$\left(\vec{\sigma}.\vec{P}\right)\left(\vec{\sigma}.\vec{P}\right) = \left(\vec{\sigma}.\vec{P}\right)^2 = \vec{P}^2 \Rightarrow H = \frac{\left(\vec{\sigma}.\vec{P}\right)^2}{2m}$$

4-1 The non-relativistic case

- To describe the particle's motion in an E.M. field the covariant derivative prescription leads to :

$$p^{\mu} \rightarrow p^{\mu} + eA^{\mu} \text{ i.e. } \begin{cases} i\hbar\partial_t \rightarrow i\hbar\partial_t + eA^0 \\ \vec{p} \rightarrow \vec{p} + e\vec{A} \end{cases}$$

- The Schrödinger equation

$$i\hbar\frac{\partial\psi}{\partial t}(\vec{x},t) = \frac{\vec{P}^2}{2m}\psi(\vec{x},t)$$

can be rewritten as

$$i\hbar\frac{\partial\psi}{\partial t}(\vec{x},t) = \frac{\left(\vec{\sigma}.\left(\vec{P}+e\vec{A}\right)\right)^2}{2m}\psi(\vec{x},t) - eA^0\psi(\vec{x},t)$$

4-1 The non-relativistic case

- Using the Pauli matrices relation :

$$\left(\vec{\sigma}.\left(\vec{P}+e\vec{A}\right)\right)^2=\left(\vec{P}+e\vec{A}\right)^2+i\vec{\sigma}.\left(\left(\vec{P}+e\vec{A}\right)\times\left(\vec{P}+e\vec{A}\right)\right)$$

and $\left(\left(\vec{P}+e\vec{A}\right)\times\left(\vec{P}+e\vec{A}\right)\right)_i=e\left(\vec{P}\times\vec{A}+\vec{A}\times\vec{P}\right)_i$

$$=e\,\varepsilon_{ijk}\left(\vec{P}_j\vec{A}_k+\vec{A}_j\vec{P}_k\right)$$

$$\Rightarrow -e\,i\hbar\varepsilon_{ijk}\left(\vec{\partial}_j\vec{A}_k+\vec{A}_j\vec{\partial}_k\right)=-e\,i\hbar\varepsilon_{ijk}\left((\vec{\partial}_j\vec{A}_k)+\underbrace{\vec{A}_k\vec{\partial}_j+\vec{A}_j\vec{\partial}_k}_{=0}\right)$$

$$=-e\,i\hbar\varepsilon_{ijk}(\vec{\partial}_j\vec{A}_k)$$

$$=-e\,i\hbar\left(\vec{\nabla}\times\vec{A}\right)_i$$

$$=e\,i\hbar\vec{B}_i$$

4-1 The non-relativistic case

- Finally the Pauli equation reads

$$i\hbar\frac{\partial\psi}{\partial t}(\vec{x},t)=\frac{\left(\vec{P}+e\vec{A}\right)^2}{2m}\psi(\vec{x},t)-\frac{e\hbar}{2m}\left(\vec{\sigma}.\vec{B}\right)\psi(\vec{x},t)-eA^0\psi(\vec{x},t)$$

where the wavefunction is a 2D object.

- The action of the spin operator is explicit.

- N.B. developing $\vec{P}+e\vec{A}$ leads to $\vec{L}.\vec{B}$. One gets the complete equation through the substitution $\vec{L}\rightarrow\vec{L}+2\vec{S}$ with $\vec{S}=\frac{\hbar}{2}\vec{\sigma}$

4-2 The Dirac equation

- Generalization to the relativistic case : the starting point is always

$$E^2 = \vec{p}^2 c^2 + m^2 c^4$$

which is the only covariant form for the canonical quantization :

$$\left[\left(i\hbar\partial_t\right)^2 - \left(\vec{\sigma}.\vec{P}\right)^2\right]\psi(\vec{x},t) = m^2\psi(\vec{x},t)$$

$$\left[i\hbar\partial_t + i\hbar\vec{\sigma}.\vec{\nabla}\right]\left[i\hbar\partial_t - i\hbar\vec{\sigma}.\vec{\nabla}\right]\psi = m^2\psi$$

- Let's take auxiliary variables :

$$\psi^{(1)} = \psi$$

$$\psi^{(2)} = \frac{1}{m}\left[i\hbar\partial_t - i\hbar\vec{\sigma}.\vec{\nabla}\right]\psi$$

4-2 The Dirac equation

- The following system

$$\begin{cases} \left[i\hbar\partial_t - i\hbar\vec{\sigma}.\vec{\nabla}\right]\psi^{(1)} = m\psi^{(2)} \\ \left[i\hbar\partial_t + i\hbar\vec{\sigma}.\vec{\nabla}\right]\psi^{(2)} = m\psi^{(1)} \end{cases}$$

is equivalent to

$$\begin{cases} i\hbar\partial_t\varphi - i\hbar\vec{\sigma}.\vec{\nabla}\chi = m\varphi \\ -i\hbar\partial_t\chi - i\hbar\vec{\sigma}.\vec{\nabla}\varphi = m\chi \end{cases} \text{ with } \begin{cases} \varphi = \psi^{(2)} + \psi^{(1)} \\ \chi = \psi^{(2)} - \psi^{(1)} \end{cases}$$

- In a matrix form this equation reads :

$$\begin{pmatrix} i\hbar\partial_t & i\hbar\vec{\sigma}.\vec{\nabla} \\ -i\hbar\vec{\sigma}.\vec{\nabla} & -i\hbar\partial_t \end{pmatrix}\begin{pmatrix} \varphi \\ \chi \end{pmatrix} = m\begin{pmatrix} \varphi \\ \chi \end{pmatrix}$$

4-2 The Dirac equation

- Gamma matrices definition :

$$\gamma^0 = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix} = \gamma_0 \text{ and } \vec{\gamma}^i = \begin{pmatrix} 0 & \sigma^i \\ -\sigma^i & 0 \end{pmatrix} = -\vec{\gamma}_i$$

- Basic properties :

$$\{\gamma^\mu, \gamma^\nu\} = \gamma^\mu \gamma^\nu + \gamma^\nu \gamma^\mu = 2g^{\mu\nu}$$

$$(\gamma^0)^2 = 1 \text{ and } (\gamma^i)^2 = -1$$

$$\gamma^{\mu\dagger} = \gamma^0 \gamma^\mu \gamma^0$$

- "Slash" notation : $\not{a} = a_\mu \gamma^\mu$

4-2 The Dirac equation

- Reminder (Pauli matrices properties) :

  - $\{\sigma_i, \sigma_j\} = 2\delta_{ij} \quad [\sigma_i, \sigma_j] = 2i\varepsilon_{ijk}\sigma_k$
  - $\sigma_i \sigma_j = \delta_{ij} + i\varepsilon_{ijk}\sigma_k$
  - $(\vec{\sigma}.\vec{O}_1)(\vec{\sigma}.\vec{O}_2) = \vec{O}_1.\vec{O}_2 + i\vec{\sigma}.(\vec{O}_1 \times \vec{O}_2)$
  - For $\vec{u}$ unit vector $\frac{1}{2}(1 \pm \vec{\sigma}.\vec{u})\chi$ eigenfunction of $\vec{\sigma}.\vec{u}$ with eigenvalue $\pm 1 \ \forall \chi$.
  - $\exp\left(-i\frac{\theta}{2}\sigma_k\right) = \cos\left(\frac{\theta}{2}\right) - i\sigma_k \sin\left(\frac{\theta}{2}\right)$

4-2 The Dirac equation

- Defining a 4D solution to the equation :

$$\psi \equiv \begin{pmatrix} \varphi \\ \chi \end{pmatrix} = \begin{pmatrix} \psi_1 \\ \psi_2 \\ \psi_3 \\ \psi_4 \end{pmatrix}$$

the previous matrix equation writes :

$$\begin{pmatrix} i\hbar\partial_t & i\hbar\vec{\sigma}.\vec{\nabla} \\ -i\hbar\vec{\sigma}.\vec{\nabla} & -i\hbar\partial_t \end{pmatrix}\begin{pmatrix} \varphi \\ \chi \end{pmatrix} = m\begin{pmatrix} \varphi \\ \chi \end{pmatrix}$$

$$\Updownarrow$$

$$i\hbar\left(\gamma^0\partial_0 + \gamma^i\partial_i\right)\psi = m\psi \text{ i.e.}$$

$$\left(i\hbar\gamma^\mu\partial_\mu - m\right)\psi = 0$$ **Dirac Equation**

- In details :

$$\sum_{\mu=0,1,2,3}\sum_{b=1,2,3,4}\left(i\hbar\left(\gamma^\mu\right)_{ab}\partial_\mu - m\delta_{ab}\right)\psi_b = 0$$

4-2 The Dirac equation

Adjoint representation.

- As usual it is obtained through the "conjugate" equation :

$$\left(i\hbar\gamma^{\mu}\partial_{\mu}-m\right)\psi=0\Rightarrow\psi^{\dagger}\left(i\hbar\gamma^{\mu\dagger}\overleftarrow{\partial}_{\mu}+m\right)=0$$

- Using the gamma matrices property : $\gamma^{\mu\dagger}=\gamma^{0}\gamma^{\mu}\gamma^{0}$ one gets

$$\psi^{\dagger}\left(i\hbar\gamma^{0}\gamma^{\mu}\gamma^{0}\overleftarrow{\partial}_{\mu}+m\gamma^{0}\gamma^{0}\right)=0$$

$$\psi^{\dagger}\gamma^{0}\left(i\hbar\gamma^{\mu}\gamma^{0}\overleftarrow{\partial}_{\mu}+m\gamma^{0}\right)=0 \qquad \leftarrow(\times\gamma^{0})$$

$$\bar{\psi}\left(i\hbar\gamma^{\mu}\overleftarrow{\partial}_{\mu}+m\right)=0 \text{ where } \boxed{\bar{\psi}=\psi^{\dagger}\gamma^{0}}$$

4-2 The Dirac equation

Adjoint representation.

- $\bar{\psi} = \psi^{\dagger}\gamma^{0}$ is the adjoint spinor

- The conjugate equations read therefore :

$$\left(i\hbar\gamma^{\mu}\partial_{\mu} - m\right)\psi = 0$$

and

$$\bar{\psi}\left(i\hbar\gamma^{\mu}\overleftarrow{\partial}_{\mu} + m\right) = 0$$

- Reminder : spinless case (K.G.)

$$\partial^{\mu}\partial_{\mu}\psi + m^{2}\psi = 0$$

and

$$\partial^{\mu}\partial_{\mu}\psi^{*} + m^{2}\psi^{*} = 0$$

4-2 The Dirac equation

Quadri-current.

- Formal derivation possibilities :

  - directly from the fundamental and adjoint representations or

  - from the Lagrangian expression ⊕ gauge invariance (looking for the current coupling to the gauge field)

- Question : is the Born probabilistic interpretation possible in the Dirac case (i.e. is the charge density positive to be interpreted as a probability density?).

- Reminder : not the case in the spinless case where negative energy solutions and negative charge density should have been re-interpretated. $\psi(x) = N e^{i(\vec{p}.\vec{x}-Et)/\hbar} \Rightarrow \rho = 2|N|^2 \times \underbrace{E}_{>0 \text{ or } 0<}$

4-2 The Dirac equation

Quadri-current.

- Direct derivation :

$$\left.\begin{array}{l}\left(i\hbar\gamma^{\mu}\partial_{\mu}-m\right)\psi=0\\ \bar{\psi}\left(i\hbar\gamma^{\mu}\overleftarrow{\partial}_{\mu}+m\right)=0\end{array}\right)\oplus$$

$$\bar{\psi}\gamma^{\mu}\left(\overleftarrow{\partial}_{\mu}+\partial_{\mu}\right)\psi=0\Leftrightarrow\partial_{\mu}\left(\bar{\psi}\gamma^{\mu}\psi\right)=0$$

- Conserved current : $\boxed{J^{\mu}\equiv c\left(\bar{\psi}\gamma^{\mu}\psi\right)}$

- Conserved charge density : $\rho\equiv J^{0}/c=\bar{\psi}\gamma^{0}\psi=\psi^{\dagger}\underbrace{\gamma^{0}\gamma^{0}}_{=1}\psi$

$$\Rightarrow\rho=\psi^{\dagger}\psi=\sum_{a=1,2,3,4}\psi_{a}^{*}\psi_{a}>0$$

4-2 The Dirac equation

Quadri-current (derived from gauge invariance).

- Lagrangian of free particles : $\mathcal{L}_{free} = \bar{\psi}\left(i\gamma^\mu \partial_\mu - m\right)\psi$

- *U(1)* invariance ➔ use of covariant derivatives :

$$\begin{aligned}\mathcal{L}_{\text{int.}} &= \bar{\psi}\left(i\gamma^\mu D_\mu - m\right)\psi \\ &= \bar{\psi}\left(i\gamma^\mu\left(\partial_\mu + ieA_\mu\right) - m\right)\psi \\ &= \bar{\psi}\left(i\gamma^\mu \partial_\mu - m\right)\psi \\ &\quad -e\underbrace{\bar{\psi}\gamma^\mu\psi}_{\propto J^\mu} A_\mu\end{aligned}$$

where the interaction term appears on the form $-J_\mu A^\mu$ as for spinless particles.

4-3 Dirac spinors

Free particle solutions.

- By analogy with the non-relativistic case we may write a general spin-½ wavefunction in terms of factorized solutions :

$$\psi = u \times (\text{plane wave}) = u \times \exp(-ip^\mu x_\mu) = u \times \exp(-ipx) = u(p)e^{\left[i(\vec{p}.\vec{x}-Et)\right]}$$

where u is a 4-components spinor.

- The explicit expression of the spinor is more easily deduced from the Hamiltonian formalism of the Dirac equation.

- The wavefunction is split into two 2-components spinors

$$\psi \equiv \begin{pmatrix} \varphi \\ \chi \end{pmatrix} \text{ where } \begin{cases} i\hbar\partial_t\varphi - i\hbar\vec{\sigma}.\vec{\nabla}\chi = m\varphi \\ -i\hbar\partial_t\chi - i\hbar\vec{\sigma}.\vec{\nabla}\varphi = m\chi \end{cases}$$

4-3 Dirac spinors

- The equations system may be written in terms of gamma matrices or equivalently :

$$\begin{cases} i\hbar\partial_t\varphi - i\hbar\vec{\sigma}.\vec{\nabla}\chi = m\varphi \\ -i\hbar\partial_t\chi - i\hbar\vec{\sigma}.\vec{\nabla}\varphi = m\chi \end{cases}$$

$$\Rightarrow\ i\hbar\partial_t\psi = \underbrace{\left(-i\hbar\left(\vec{\alpha}.\vec{\nabla}\right) + \beta m\right)}_{H_D}\psi$$

$$\text{where } \alpha_i = \begin{pmatrix} 0 & \sigma_i \\ \sigma_i & 0 \end{pmatrix} \text{ and } \beta = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}$$

- The eigenfunctions are solutions of :

$$H_D u = \left(\vec{\alpha}.\vec{p} + \beta m\right)u = Eu$$

4-3 Dirac spinors

- Assuming the spinor u into two 2-components spinors u_A and u_B

$$u \equiv \begin{pmatrix} u_A \\ u_B \end{pmatrix}$$

$$\Rightarrow H_D u = (\vec{\alpha}.\vec{p} + \beta m) u = \begin{pmatrix} m \times 1 & \vec{\sigma}.\vec{p} \\ \vec{\sigma}.\vec{p} & -m \times 1 \end{pmatrix} \begin{pmatrix} u_A \\ u_B \end{pmatrix} = E \begin{pmatrix} u_A \\ u_B \end{pmatrix}$$

$$\Rightarrow \begin{cases} (\vec{\sigma}.\vec{p}) u_B = (E - m) u_A \\ (\vec{\sigma}.\vec{p}) u_A = (E + m) u_B \end{cases}$$

- The system leads to :

  - a generic form for u : $u = \begin{pmatrix} u_A \\ \dfrac{\vec{\sigma}.\vec{p}}{E+m} u_A \end{pmatrix}$

  - a relation between E and $\boldsymbol{p}$:

$$(\vec{\sigma}.\vec{p})(\vec{\sigma}.\vec{p}) u_A = \vec{p}^2 u_A = (E - m)(E + m) u_A \Rightarrow E = \pm E_p$$

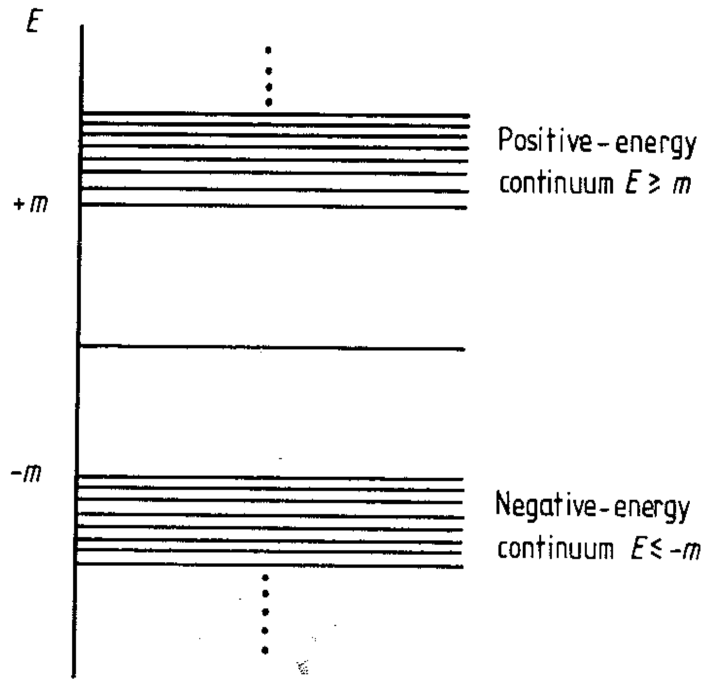
4-3 Dirac spinors

Summary.

- Klein-Gordon equation ➔ { negative probabilities; negative energies }
- Dirac equation ➔ { positive (only) probabilities; negative energies }
- Dirac's historical interpretation : the 'vacuum' state consists of all negative-energy states filled with electrons. The Pauli principle forbids any positive-energy electron from falling into these lower energy states.

4-3 Dirac spinors

- The 'vacuum' (so-called Dirac sea) has now infinite negative charge and energy but all observations represent finite fluctuations w.r.t. the vacuum.



- A 'hole' in the Dirac sea, i.e. the absence of a negative-energy electron is equivalent to the presence of a positive-energy positively charged version of the electron, namely a positron.

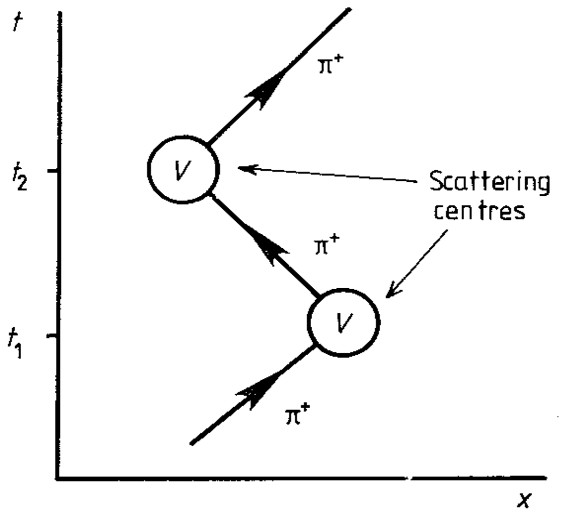
$$\text{energy of 'hole'} = -\left(E_{neg}\right) \rightarrow \text{positive energy}$$

$$\text{charge of 'hole'} = -\left(q_e\right) \rightarrow \text{positive charge}$$

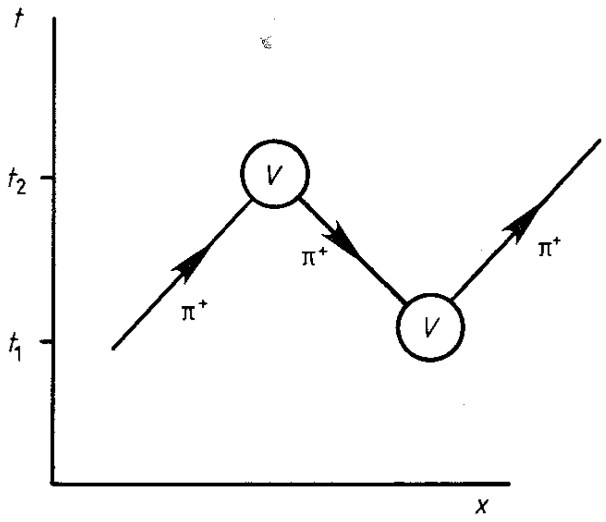
4-3 Dirac spinors

- What about solutions to the Klein-Gordon equation (bosons not affected by Pauli principle)?
- Interpretation through Feynman's prescription (1962) :

negative-energy particle solutions propagating backward in time $\equiv$
positive-energy antiparticle solutions propagating forward in time



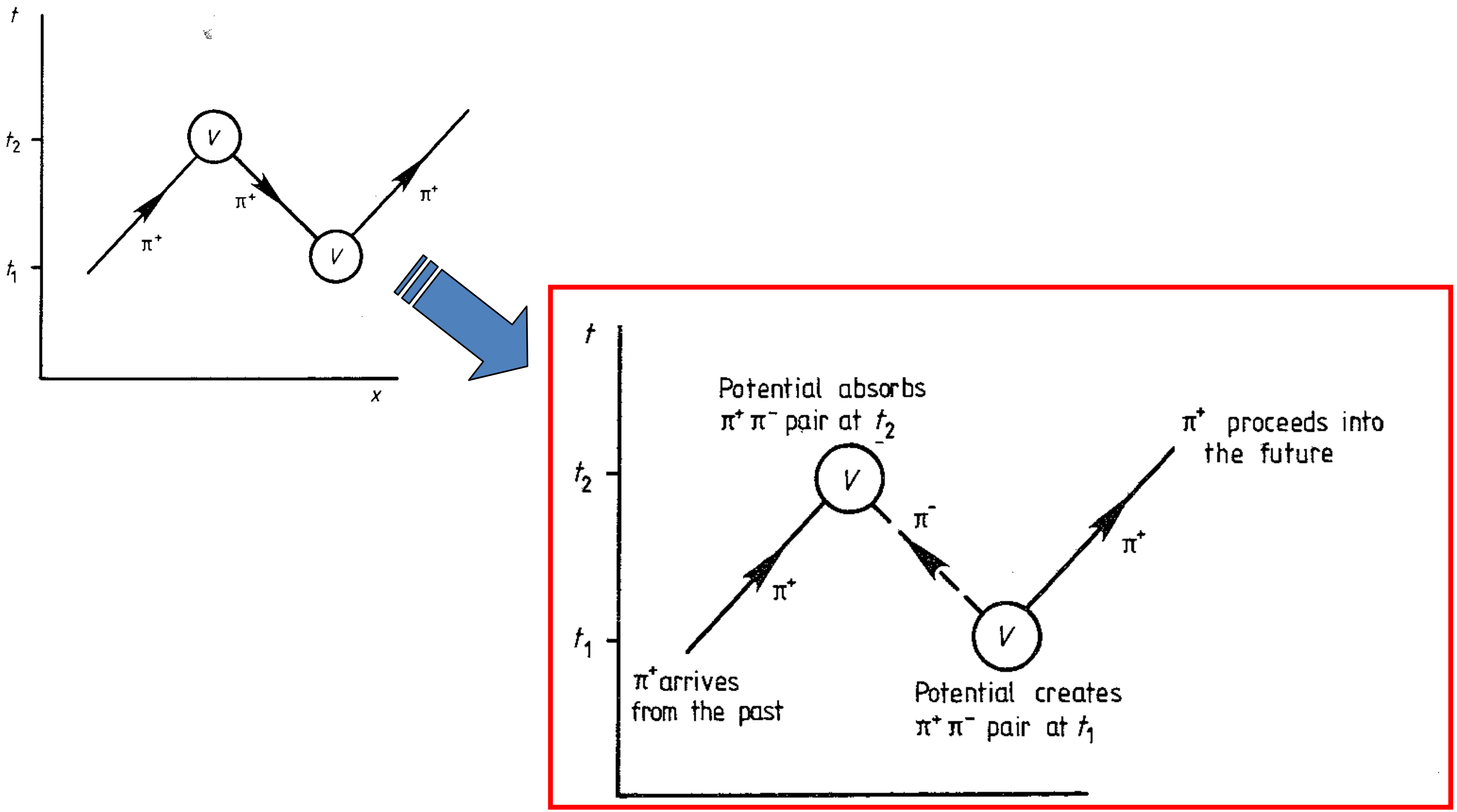
In NRQM



To be added in RQM

4-3 Dirac spinors

- Consequence of the Feynman prescription :



4-3 Dirac spinors

Summary.

- Positive-energy solutions :

$$\psi^{(+)(s=1,2)}(x) \equiv u^{(s=1,2)}(p)\, e^{[-ipx]} \quad u^{(s)}(p) \propto \begin{pmatrix} \varphi^{(s)} \\ \dfrac{\vec{\sigma}.\vec{p}}{E+m}\varphi^{(s)} \end{pmatrix}$$

- Negative-energy solutions :

$$\psi^{(-)(s=1,2)}(x) \equiv v^{(s=1,2)}(p)\, e^{[+ipx]} \quad v^{(s)}(p) \propto \begin{pmatrix} \dfrac{\vec{\sigma}.\vec{p}}{E+m}\chi^{(s)} \\ \chi^{(s)} \end{pmatrix}$$

with the spin-up/down 2D spinors $\varphi,\chi^{(s=1)} = \begin{pmatrix} 1 \\ 0 \end{pmatrix}$ and $\varphi,\chi^{(s=2)} = \begin{pmatrix} 0 \\ 1 \end{pmatrix}$

- Warning on spin direction : negative-energy solution with spin-↑ = positive-energy solution with spin-↓

4-3 Dirac spinors

- Dirac equation in *p*-space :

$$\left(\gamma^\mu p_\mu - m\right)u(p) = 0$$
$$\left(\gamma^\mu p_\mu + m\right)v(p) = 0$$

- Adjoint spinors (same procedure as in x-space) :

$$\bar{u}(p)\left(\gamma^\mu p_\mu - m\right) = 0$$
$$\bar{v}(p)\left(\gamma^\mu p_\mu + m\right) = 0$$

- Ortho-normalization relations :

$$\bar{u}^{(s)}(p)u^{(s')}(p) = \delta_{ss'} \qquad \bar{v}^{(s)}(p)v^{(s')}(p) = -\delta_{ss'}$$
$$\bar{u}^{(s)}(p)v^{(s')}(p) = 0 \qquad \bar{v}^{(s)}(p)u^{(s')}(p) = 0$$

4-4 Spin and Energy projectors

Some useful relations :

$$\sum_{s=1,2} u^{(s)}(p)\bar{u}^{(s)}(p) = (\not{p} + m)$$

$$\sum_{s=1,2} v^{(s)}(p)\bar{v}^{(s)}(p) = (\not{p} - m)$$

- 'Energy' projectors :

$$\Lambda_+ = \frac{(\not{p} + m)}{2m} \text{ such as } \Lambda_+ u = u \text{ and } \Lambda_+ v = 0$$

$$\Lambda_- = \frac{(-\not{p} + m)}{2m} \text{ such as } \Lambda_- v = v \text{ and } \Lambda_- u = 0$$

$$\Lambda_\pm^2 = \Lambda_\pm \text{ and } \Lambda_+ + \Lambda_- = 1$$

4-4 Spin and Energy projectors

- Spin projectors. The projection is done along the propagation direction of the particle (helicity operator) chosen as the 3rd axis :

$$\vec{p} = p_z \vec{e}_z \Rightarrow \frac{\vec{S}.\vec{p}}{|\vec{p}|} = S_z = \frac{\hbar}{2}\begin{pmatrix} \sigma_3 & 0 \\ 0 & \sigma_3 \end{pmatrix} = \frac{\hbar}{2}\begin{pmatrix} 1 & & & \\ & -1 & & \\ & & 1 & \\ & & & -1 \end{pmatrix}$$

- $s=1$

$$u^{(s=1)}(p) = N\begin{pmatrix} \begin{pmatrix} 1 \\ 0 \end{pmatrix} \\ \frac{\vec{\sigma}.\vec{p}}{E+m}\begin{pmatrix} 1 \\ 0 \end{pmatrix} \end{pmatrix} = N\begin{pmatrix} \begin{pmatrix} 1 \\ 0 \end{pmatrix} \\ \frac{|\vec{p}|}{E+m}\begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}\begin{pmatrix} 1 \\ 0 \end{pmatrix} \end{pmatrix} = N\begin{pmatrix} \begin{pmatrix} 1 \\ 0 \end{pmatrix} \\ \frac{|\vec{p}|}{E+m}\begin{pmatrix} 1 \\ 0 \end{pmatrix} \end{pmatrix}$$

$$v^{(s=1)}(p) = N\begin{pmatrix} \frac{\vec{\sigma}.\vec{p}}{E+m}\begin{pmatrix} 1 \\ 0 \end{pmatrix} \\ \begin{pmatrix} 1 \\ 0 \end{pmatrix} \end{pmatrix} = N\begin{pmatrix} \frac{|\vec{p}|}{E+m}\begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}\begin{pmatrix} 1 \\ 0 \end{pmatrix} \\ \begin{pmatrix} 1 \\ 0 \end{pmatrix} \end{pmatrix} = N\begin{pmatrix} \frac{|\vec{p}|}{E+m}\begin{pmatrix} 1 \\ 0 \end{pmatrix} \\ \begin{pmatrix} 1 \\ 0 \end{pmatrix} \end{pmatrix}$$

4-4 Spin and Energy projectors

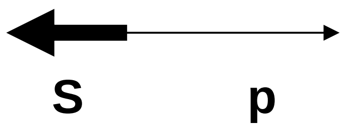
- *s=2*

$$u^{(s=2)}(p) = N\begin{pmatrix} \begin{pmatrix} 0 \\ 1 \end{pmatrix} \\ \dfrac{-|\vec{p}|}{E+m}\begin{pmatrix} 0 \\ 1 \end{pmatrix} \end{pmatrix} \text{ and } v^{(s=2)}(p) = \begin{pmatrix} \dfrac{-|\vec{p}|}{E+m}\begin{pmatrix} 0 \\ 1 \end{pmatrix} \\ \begin{pmatrix} 0 \\ 1 \end{pmatrix} \end{pmatrix}$$

- The spinors are eigenstates of *Sz* with eigenvalues ±½

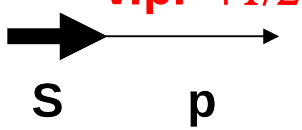
$$S_z u^{(s)}(p) = \pm\frac{\hbar}{2}u^{(s)}(p) \text{ for } s = \begin{cases} 1 \\ 2 \end{cases}$$

v.p.=–1/2



Left-handed

v.p.=+1/2



Right-handed

4-4 Spin and Energy projectors

- Case of massless fermions :

$$H_D u = (\vec{\alpha}.\vec{p})u = \begin{pmatrix} 0 & \vec{\sigma}.\vec{p} \\ \vec{\sigma}.\vec{p} & 0 \end{pmatrix} \begin{pmatrix} u_A \\ u_B \end{pmatrix} = E \begin{pmatrix} u_A \\ u_B \end{pmatrix} \Rightarrow \begin{cases} (\vec{\sigma}.\vec{p})u_B = Eu_A \\ (\vec{\sigma}.\vec{p})u_A = Eu_B \end{cases}$$

one has two solutions : $u_B = u_A$ or $u_B = -u_A$

- Dirac equation decouples into two equations for 2D spinors :

$$\begin{cases} (\vec{\sigma}.\vec{p})\omega = -E\omega & (1) \\ (\vec{\sigma}.\vec{p})\eta = +E\eta & (2) \end{cases} \text{ with } E = \pm|\vec{p}|$$

- positive-energy solution for (1) => $(\vec{\sigma}.\hat{p})\omega = -\omega$

describes a neutrino of energy *E* and negative helicity => left ν

- negative-energy solution for (1) => $(\vec{\sigma}.\hat{p})\omega = +\omega$

describes an antineutrino of energy *E* and positive helicity => right $\overline{\nu}$

4-4 Spin and Energy projectors

Helicity projector :

- Case of massless fermions (II) :

$$(1) \Rightarrow \begin{cases} \nu_L \\ \bar{\nu}_R \end{cases} \text{ and } (2) \Rightarrow \begin{cases} \nu_R \\ \bar{\nu}_L \end{cases}$$

- Defining the matrix :

$$\gamma_5 = i\gamma_0\gamma_1\gamma_2\gamma_3 = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} \Rightarrow \begin{cases} \{\gamma_5, \gamma^\mu\} = 0 \\ \gamma_5^2 = 1 \end{cases}$$

one uses the following projectors onto L or R states to generate the electron-neutrino coupling term (V-A theory of weak interactions) :

$$J^\mu = \bar{\psi}_e \gamma^\mu \frac{1}{2}(1 - \gamma_5)\psi_{\nu_e}$$

4-5 Charge conjugation operator

A ½-spin particle in an E.M. field follows the usual Dirac equation :

$$\left[\gamma^{\mu}\left(i\partial_{\mu}-eA_{\mu}\right)-m\right]\psi=0$$

What is the analog for an anti-particle described by a C-conjugate spinor?

$$\left[\gamma^{\mu}\left(i\partial_{\mu}+eA_{\mu}\right)-m\right]\psi_{C}=0$$

Following basic developments : $\left[\gamma^{\mu}\left(i\partial_{\mu}-eA_{\mu}\right)-m\right]\psi=0$

$$\Rightarrow\left[\gamma^{\mu*}\left(-i\partial_{\mu}-eA_{\mu}\right)-m\right]\psi^{*}=0$$

$$\Rightarrow\left[-\gamma^{\mu*}\left(i\partial_{\mu}+eA_{\mu}\right)-m\right]\psi^{*}=0$$

Defining C such that : $-\left(C\gamma^{0}\right)\gamma^{\mu*}=\gamma^{\mu}\left(C\gamma^{0}\right)$ one gets

$$\left[\gamma^{\mu}\left(i\partial_{\mu}+eA_{\mu}\right)-m\right]\left(C\gamma^{0}\psi^{*}\right)=0 \text{ i.e. } \boxed{\psi_{C}=C\gamma^{0}\psi^{*}}$$

4-5 Charge conjugation operator

In Dirac representation the C-conjugation operator is given by :

$$\psi_C = C\gamma^0\psi^* \Rightarrow C = i\gamma^2\gamma^0\hat{K} = C^{-1}$$

Application :

$$\left(\psi^{(+)(s)}\right)_C = i\gamma^2\left(u^{(s)}(p)\,e^{[-ipx]}\right)^* = i\begin{pmatrix} 0 & \sigma_2 \\ -\sigma_2 & 0 \end{pmatrix}\begin{pmatrix} \chi^{(s)} \\ \frac{\vec{\sigma}^*.\vec{p}}{E+m}\chi^{(s)} \end{pmatrix} e^{[ipx]}$$

$$= i\begin{pmatrix} \frac{\sigma_2\vec{\sigma}^*.\vec{p}}{E+m}\chi^{(s)} \\ -\sigma_2\chi^{(s)} \end{pmatrix} e^{[ipx]} = i\begin{pmatrix} \frac{-\vec{\sigma}.\vec{p}\sigma_2}{E+m}\chi^{(s)} \\ -\sigma_2\chi^{(s)} \end{pmatrix} e^{[ipx]}$$

$$= \begin{pmatrix} \frac{\vec{\sigma}.\vec{p}}{E+m}\chi^{(s')} \\ \chi^{(s')} \end{pmatrix} e^{[ipx]} = -v^{(s')}(p)\,e^{[ipx]} = -\psi^{(-)(s')}$$

Summary

- Relativistic propagation equations derived from energy definition law => Klein-Gordon (bosons) and Dirac (fermions)

- Gauge invariance imposed to the theory => existence of conserved currents coupled to gauge fields (e.g. photons for EM)

- From classical to quantum field theory : fields => operators

- Lagrangian formalism widely used.