Chapter 4

Introduction to gauge theory

Outline/Plan

1. Introduction
2. Lagrangian formalism
3. Gauge invariance
4. U(1) gauge field

1. Introduction
2. Formalisme Lagrangien
3. Invariance de jauge
4. Champ de jauge U(1)

1- Introduction

- Particle physics relies on quantum field theory which is commonly expressed in Lagrangian formalism.

- Reminder :

  - In classical mechanics the particle's motion is described by the Lagrange equations:

$$\frac{d}{dt}\left(\frac{\partial L}{\partial \dot{q}_i}\right) - \frac{\partial L}{\partial q_i} = 0$$

where q_i are the generalized coordinates of the particles and $\dot{q}_i = \frac{dq_i}{dt}$ their time derivatives.

  - The Lagrangian of the system is defined as $L = T - V$ where T and V are the kinetic and potential energies.

2- Lagrangian formalism

- From discrete to continuous variables $\psi(\vec{x},t)$:

  - the Lagrangian is replaced by a Lagrangian density

$$L(q_i,\dot{q}_i,t) \to \mathcal{L}\left(\psi,\partial_\mu\psi,x_\mu\right)$$

  - the normalization of the Lagrangian density is such that :

$$L = \int d^3x \mathcal{L}$$

  - the Euler-Lagrange equations read :

$$\partial_\mu\left(\frac{\partial\mathcal{L}}{\partial\left(\partial_\mu\psi\right)}\right) - \frac{\partial\mathcal{L}}{\partial\psi} = 0$$

- Starting from the Lagrangian density one defines an action :

$$S(\psi) = \int d^4x \mathcal{L}\left(\psi,\partial_\mu\psi,x_\mu\right)$$

2- Lagrangian formalism

- Noether's theorem : each invariance of the theory (Lagrangian density) implies the conservation of a charge and a current

- For instance the variation of the action $S' = S(\psi')$ expressed in terms of the transformed fields ψ' under a local transformation depending on the parameter $\alpha(x)$ reads :

$$\delta S = S' - S = \int d^4x\, \alpha(x)\, \partial_\mu J^\mu$$

- The least action principle leads to the continuity equation :

$$\partial_\mu J^\mu = 0$$

describing the conservation of the charge $Q = \int d^3x\, J^0$

2- Lagrangian formalism

- The physics of a given type of particle is described through a Lagrangian density involving quantum fields which can be seen as creation/annihilation operators of particles in the standard of 2nd quantization.

- For example the free movement of a spinless particle is described by the following Lagrangian density:

$$L_{free} = \partial_\mu \psi^\dagger \partial^\mu \psi - m^2 \psi^\dagger \psi$$

Applying the Euler-Lagrange equations $\partial_\mu \left(\frac{\partial L}{\partial \left(\partial_\mu \psi^\dagger \right)} \right) - \frac{\partial L}{\partial \psi^\dagger} = 0$

we get the K-G equation : $\partial_\mu \left(\partial^\mu \psi \right) - (-m^2 \psi) = 0$

3- Gauge invariance

- The invariance of the theory (Lagrangian) under a
  - space translation
  - time translation
  - space rotation

  is associated with the conservation of $\vec{p},\ E,\ \vec{J}$

- Those symmetries are "space-time" like. The theory can be also invariant under internal symmetries. For instance for an electron described by a field ψ the Lagrangian is invariant under the global phase transform :

$$\psi \to e^{i\alpha}\ \psi$$

The transforms $U\left(\alpha\right) = e^{i\alpha}$ constitute the Abelian group $U\left(1\right)$

3- Gauge invariance

- The conserved quantity in that case corresponds to the electrical charge.
- Starting from an infinitesimal transform $\psi \rightarrow (1+i\alpha)\psi$ one derives the real form of the conserved current…
- Physically the existence of a symmetry implies that a quantity is not observable (e.g. the invariance under a space translation means that it is not possible to fix an absolute position in space which can be therefore chosen arbitrarily).
- In the $U(1)$ case the quantity α is called a global gauge.

In the particle physics Standard Model the fundamental interactions are built on symmetry principles, those of the ***local gauge*** transforms.

3- Gauge invariance

Local gauge invariance of the free particles Lagrangian :

- under the local transform $\psi \to e^{i\alpha}\,\psi$ where the α parameter depends on x^μ the Lagrangian

$$L_{free} = \partial_\mu \psi^\dagger \partial^\mu \psi - m^2 \psi^\dagger \psi$$

is not invariant :

$$\psi \to e^{i\alpha}\,\psi \quad \Rightarrow \partial^\mu \psi \to \left(\partial^\mu + i\left(\partial^\mu \alpha\right)\right)\psi$$

$$\psi^\dagger \to e^{-i\alpha}\,\psi^\dagger \quad \Rightarrow \partial^\mu \psi^\dagger \to \left(\partial^\mu - i\left(\partial^\mu \alpha\right)\right)\psi^\dagger$$

$$\psi^\dagger \psi \to \psi^\dagger \psi \quad ☺$$

$$\partial_\mu \psi^\dagger \partial^\mu \psi \to \left(\partial^\mu - i\left(\partial^\mu \alpha\right)\right)\psi^\dagger \left(\partial^\mu + i\left(\partial^\mu \alpha\right)\right)\psi$$

$$= \partial_\mu \psi^\dagger \partial^\mu \psi - i\left(\partial^\mu \alpha\right)\psi^\dagger \partial^\mu \psi + \psi^\dagger i\left(\partial^\mu \alpha\right)\psi \ldots \quad ☹$$

3- Gauge invariance

Local gauge invariance of the free particles Lagrangian :

- To force the invariance one introduces a covariant derivative :

$$D^{\mu}\psi = \left(\partial^{\mu} - ieA^{\mu}\right)\psi$$

where A^{μ} is the gauge field which should transform as :

$$A^{\mu} \rightarrow A^{\mu} + \frac{1}{e}\partial^{\mu}\alpha$$

in order to balance the transform of the derivative terms

$$D^{\mu}\psi \rightarrow \left(\partial^{\mu} + i\left(\partial^{\mu}\alpha\right) - ieA^{\mu} - ie\left(\frac{1}{e}\partial^{\mu}\alpha\right)\right)\psi$$

$$D^{\mu}\psi^{\dagger} \rightarrow \left(\partial^{\mu} - i\left(\partial^{\mu}\alpha\right) + ieA^{\mu} + ie\left(\frac{1}{e}\partial^{\mu}\alpha\right)\right)\psi^{\dagger}$$

3- Gauge invariance

Local gauge invariance of the free particles Lagrangian :

- The Lagrangian is modified into :

$$\begin{aligned} L_{\text{int}} &= D_\mu \psi^\dagger D^\mu \psi - m^2 \psi^\dagger \psi \\ &= \partial_\mu \psi^\dagger \partial^\mu \psi - m^2 \psi^\dagger \psi \\ &+ ie\left[A_\mu \psi^\dagger \partial^\mu \psi - \partial_\mu \psi^\dagger A^\mu \psi \right] \\ &+ e^2 A_\mu \psi^\dagger A^\mu \psi \end{aligned}$$

- Forcing the *U(1)* invariance introduces a new vector field, the gauge field, which couples to the particles through 2 different types of vertices. Generic coupling term: $-J_\mu A^\mu$

- This gauge field is associated to the photon, responsible for the electromagnetic interaction.

4- U(1) gauge field

To really associate the gauge field of the U(1) symmetry to the photon it is mandatory to include the dynamics of the photon itself:

- Propagation equation in vacuum : $\Box A^{\mu} = 0$
- Photon interaction with its sources (Maxwell equations) :

$$\partial_{\mu} F^{\mu\nu} = J^{\nu}$$

with the field tensor : $F^{\mu\nu} = \partial^{\mu} A^{\nu} - \partial^{\nu} A^{\mu}$

- Reminder : classical fields

$$A^{\mu} = \left(V, \vec{A}\right)$$

$$F_{0i} = \partial_0 A_i - \partial_i A_0 = -\frac{\partial\left(\vec{A}\right)_i}{\partial t} - \left(\vec{\nabla}\right)_i V = \left(\vec{E}\right)_i$$

$$F_{ij} = \partial_i A_j - \partial_j A_i = \vec{\nabla}_i \vec{A}_j - \vec{\nabla}_j \vec{A}_i = \varepsilon_{ijk} \left(\vec{\nabla} \times \vec{B}\right)_k = \varepsilon_{ijk} \vec{B}_k$$

4- U(1) gauge field

- Propagation equations of the interacting field :

$$\partial_\mu F^{\mu\nu} = J^\nu$$

$$\Downarrow$$

$$\partial_\mu \left(\partial^\mu A^\nu - \partial^\nu A^\mu \right) = J^\nu$$

$$\Box A^\nu - \partial^\nu \underbrace{\left(\partial_\mu A^\mu \right)}_{\substack{=0 \\ \text{in Lorentz} \\ \text{gauge}}} = J^\nu \Rightarrow \boxed{\Box A^\nu = J^\nu}$$

- Dynamic term for the photon

$$\mathcal{L}_{free} = -\frac{1}{4} F_{\mu\nu} F^{\mu\nu} \quad \Bigg| \quad \partial_\mu \left(\frac{\partial \mathcal{L}}{\partial \left(\partial_\mu A_\nu \right)} \right) - \frac{\partial \mathcal{L}}{\partial A_\nu} = 0$$

- No mass term (!) : $m^2 A_\mu A^\mu$ not invariant under *U(1)* transform

4- U(1) gauge field

Summary :

- Particle physics is described by local gauge theories
- Each invariance is associated with gauge field(s)
- Gauge fields couple to the particles
- For spinless particles the Q.E.D. Lagrangian density reads :

$$
\begin{aligned}
L_{spin-0} = \; & \partial_\mu \psi^\dagger \partial^\mu \psi - m^2 \psi^\dagger \psi \\
& + ie\left[A_\mu \psi^\dagger \partial^\mu \psi - \partial_\mu \psi^\dagger A^\mu \psi \right] + e^2 A_\mu \psi^\dagger A^\mu \psi \\
& - \frac{1}{4} F_{\mu\nu} F^{\mu\nu}
\end{aligned}
$$

- What about ½-spin particles description?